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DECOMPOSITION AND INTERCONNECTED SYSTEMS IN MATHEMATICAL PROGRAMMING

by
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CHAPTER I

INTRODUCTION

1-1. Objectives and Summary.

This thesis will be essentially devoted to the development of a decomposition method for solving large linear programming problems arising from N interconnected linear systems. These problems, which will be referred to as linear interconnection problems, can be stated as follows:[1]

(1) Minimize $Z = cx$ subject to $Ax = b$, $x \geq 0$ where the coefficient matrix A has a structure of the type

(2)

A^1			I	-I			I	-I
	A^2		-I	I	I	-I		
		A^3			-I	I	-I	I

We point out immediately that the mulstistage linear problem concerned with dynamic situations having a finite number of time periods is a particular form of problem (1), whose coefficient matrix has the staircase structure

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[1] The notations adopted in this report are explained in section 2 of this chapter.

$$
(3) \qquad \begin{array}{|c|c|c|c|c|}
\hline
A^1 & -I & & & \\
\hline
 & I & A^2 & -I & \\
\hline
 & & & I & A^3 \\
\hline
\end{array}
$$

where the matrices A^1, A^2 and A^3 might be identical.

Linear programs having a coefficient matrix similar to matrix (2) might represent, for instance, optimization problems arising from the cooperation of three economies (or industries) whose production functions are assumed to be linear. In such a case the matrices A^1, A^2 and A^3 would represent input-output coefficient matrices of economies 1, 2 and 3 respectively, whereas the connection matrices I and -I would represent importation and exportation matrices. The objective function, in this case, could be thought of as a common social welfare function [32, p. 47] which has to be optimized.

In this report we first develop two decomposition methods for solving the special class of linear interconnection problems for which the above matrices A^1, A^2 and A^3 are Leontief matrices. Although the solution methods for this type of problem might have a wide range of application -- notably to the study of Leontief economic models and sequential decision problems [16] -- it is not the practical usefulness of these models which motivated their extensive development in this report, but rather the fact that, as far as mathematical programming is concerned, they have an ideal behavior which gives us valuable insight into more complex problems.

In chapter II we introduce the simplest decomposition approach which one might conjecture would work, i.e., a method based on communication of prices only. We refer to this method as a <u>price-communication decomposition method</u>. In chapter III, which is the core of this study, we develop a decomposition method for solving linear programming problems of type (2) for interconnected Leontief systems. This method, which is called decomposition method by square-block triangularization, is extended in chapter IV to general interconnected matrices. The essential ideas underlying this decomposition method are the following.

Given a basic feasible solution to a linear program of type (1) we shall transform this problem into an equivalent linear program which will have a feasible basis in the following square block triangular form

$$B = \begin{array}{|c|c|c|} \hline B^1 & E^1 & E^2 \\ \hline 0 & B^2 & E^3 \\ \hline 0 & 0 & B^3 \\ \hline \end{array}$$

where the submatrices B^1, B^2 and B^3 are square matrices. Next, we shall see that we can find an improved solution of this equivalent problem by solving, via the simplex method, a smaller subproblem called the improvement subprogram. If this improved solution is not optimal, we set up another equivalent problem which, in its turn, will lead to another improvement subprogram, and so on, until optimality is reached.

The main features of the two decomposition procedures just mentioned are the following:

(a) All the coefficient matrices of the improvement subprograms have the following structure:

$$\begin{array}{|c|c|c|c|}\hline A^i & T^1 & \cdots & T^n \\ \hline \end{array}$$

where

A^i is the submatrix of A corresponding to the i^{th} system;

T^j (for $j \neq i$ only) is a modified exchange matrix of the form

$$T^j = I_m + \theta^j = \begin{array}{|ccccc|}\hline 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ \hline X & \cdots & X & X & X \\ X & & X & \cdots & X \\ \hline \end{array}$$

Where I_m is the identity matrix of order m.

(b) The exchange of information which is required between the subprograms consists of

(i) the simplex multipliers only for the price communication decomposition algorithm of chapter II;

(ii) the simplex multipliers and the matrices θ^j for the square-block triangularization decomposition algorithm when applied to Leontief systems;

(iii) the simplex multipliers, the matrices θ^j plus some additional information related to the feasibility of the solutions when applied to general systems.

Remark. At this point it should be mentioned that when the decomposition principle of Dantzig and Wolfe [9, ch. 23], [14] and [15] is applied to solve problem (1), its master program behaves to some extent like an improvement subprogram of the square block triangularization decomposition method, whereas its subproblems are rather similar to those obtained by the price communication decomposition method.

(c) The decomposition procedures considered in this report are symmetric in the sense that

(i) all subprograms have the same structure;

(ii) the exchange of information between the subprograms is the same in all directions.

This feature might be of importance when a horizontal decentralization of the decision making process becomes necessary. By this we mean that no hierarchy exists between the interconnected systems. For instance, problems concerning the cooperation of several independent economies would certainly require this type of decentralization.

(d) The decomposition methods are always primal feasible.

By this we mean that, at any stage of the decomposition, basic feasible solutions to the original problem as well as the subproblems are known.

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The idea of taking advantage of the square block triangularity of the bases is, of course, not new. In particular, Dantzig [11] has applied it to solve dynamic Leontief systems with substitution, and has even suggested an interesting decomposition method for solving multi-stage problems which is based on artificial square block triangular bases, [10] and [12]. However, the computations required by this last method appeared quite complex, and it seems that it has not attracted much attention, especially since the discovery of the decomposition principle of Dantzig and Wolfe which promised much more elegant solutions for these multistage problems.

Then, in recent years, came the development of methods which are basically variants of the simplex method adapted to special structures of large scale linear programming problems. All these methods take advantage of the observation that "the inverse of the basis in the simplex method serves no function except as a means for obtaining the representation of the vector entering the basis and for determining the new price vector" [13, p. 1]. Among these methods we mention for their approach the primal partitioning programming procedure, Rosen [34]; the pseudo-basic variable procedure, Beale [6]; the compact basis triangularization method, Dantzig [13]; and very recently, the dualplex method, Gass [23].

We must also mention that Abadie [3] has shown that, under certain conditions, if the decomposition method of Dantzig and Wolfe is specialized to obtain basic solutions at each iteration, then it becomes nothing else than a variant of the simplex method for which the inverse of the current basis is computed by partitioning. We believe that this observation is important because it indicates that, provided an adequate exchange of information takes place, the primal simplex method lends itself to decentralized computations. Since there is no good reason to think that the simplex method which has proved to be very efficient for the solution of small linear programs will not be as well adapted to the solution of large ones, it seems that efficient decomposition methods for solving the latter might well be sought within the framework of the simplex method.

To conclude these remarks we observe that the main factors which have to be considered in the choice of a decomposition procedure are

(a) the structure of the basis;

(b) the partitioning imposed (if any) by the necessity of decentralization;

(d) the type of exchange of information between the subprograms which is desirable.

1-2. Terminology and Notations.

In this report we shall adhere most of the time to the standard terminology in mathematical programming, for which we refer to [9], [21], [22], and [26], and for the rest we shall define the technical terms and symbols as they are introduced. Therefore, only a few notational remarks are in order here.

We first give a list of the principal notations used in this report to best illustrate the conventions we have adopted.

A, B, C, D, E, H, P, Q, R, S, T denote exclusively matrices.

$|A|$ denotes the determinant of the square matrix A.

$A_{.j}$ denotes the j^{th} column vector of the matrix A.

$A_j = A_{.j}$ if no confusion is possible.

$A_{i.}$ denotes the i^{th} row vector of the matrix A.

A_J denotes the submatrix of the $(m \times n)$ matrix A whose columns are $(A_j)_{j \in J}$ where $J \supset N = \{1, 2, \ldots, n\}$.

$A_{\bar{J}}$ denotes the set of columns of A which are not in A_J

α, λ, τ, μ, ν normally denote scalars.

b, c, d, x, t, w, y denote vectors. Note that no distinction is made between row and column vectors.

b_J denotes the vector whose components are (b_i), $i \in J$.

B normally denotes a feasible basis of a linear program.

β always denotes the inverse of the matrix B.

e_m = (1, 1, . . . , 1) denotes the m-component vector all of whose components are one.

u_i = (0, . . . 0, 1, 0. . . 0) denotes the i^{th} unit vector whose i^{th} coordinate is one and whose other coordinates are zero.

I_m denotes the identity matrix of order m.

I^i, J^i, K denote sets of positive integers (indices).

M = {1, 2, . . . m} denotes the set of positive integers from 1 to m.

N_i = {1, 2, . . . n_i} denotes the set of positive integers from 1 to n_i.

ϕ denotes the empty set.

π always denotes the price vector (simplex multipliers) associated with a basis B.

γ always denotes the cost vector associated with a basis B. By definition we have $\pi B = \gamma$.

w always denotes a vector of basic variables associated with a basis B. By definition $Bw = b$.

J = {j/x_j is a basic variable} denotes the set of basic activities; therefore, $w = x_J$.

Z always represents the value of the objective function.

Unless otherwise specified, these notations will be consistently used, and often without further explanation.

It remains to indicate briefly how the indices will be used. For an iterative process we will adjoin an argument t to any quantity which might vary. Thus, the basis B of a linear program at the t^{th} iteration will be denoted by B(t), the simplex multipliers by $\pi(t)$ and the values of the basic solutions by w(t). The next feature which requires distinctive notations is the decomposition of a linear program into smaller subprograms. In this case, all quantities relative to the i^{th} subprogram will have a superscript i, e.g. $B^i(t)$, $\pi^i(t)$, $w^i(t)$ which are not to be confused either with the i^{th} column of B(t), i.e., $B_i(t)$, or with the i^{th} components of $\pi(t)$ and $w(t)$ $\left(\pi_i(t) \text{ and } w_i(t)\right)$.

Finally, a word about our numbering system. Theorems, definitions, remarks as well as equations are all numbered consecutively within each chapter. A reference to an equation outside a given chapter will be made by prefixing the chapter number to the equation number. The numbers in square brackets refer to books and papers listed in the bibliography at the end of this report.

CHAPTER II

OPTIMIZATION BY PRICE COMMUNICATION BETWEEN LEONTIEF SYSTEMS[1]

2-1. Introduction.

In this chapter we shall be concerned with large linear programming problems based on interconnected Leontief systems whose activity levels are unbounded and whose coefficient matrices have a form similar to

A^1			I	I	-I		-I	
	A^2		-I		I	I		-I
		A^3		-I		-I	I	I

where A^1, A^2 and A^3 are Leontief matrices with substitution whose precise definition will be given in the next section

We shall determine some important properties of this class of problems by concentrating, in the following sections, on a decomposition procedure based exclusively on price communication between the Leontief systems. It should be noted that this decomposition procedure has only a limited practical interest for the following reasons:

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[1] Basically, the ideas developed in this chapter were suggested by Prof. G. B. Dantzig during a seminar on Computational Methods in Mathematical Programming which was given during the fall semester 1962 at the University of California, Berkeley.

(i) we prove only that an optimal basis to the original problem can be found in a finite number of iterations;

(ii) no optimality criterion is given;

(iii) a better convergence to the optimal basis can be achieved by another method described in chapter 3.

Nevertheless, as will be illustrated in chapter 3, we remark that for the first iterations the above method could give good results and, therefore, might be efficiently used to find an improved initial basis to a more complex decomposition approach. Also, it might be advantageous to use the price communication decomposition algorithm in certain cases where only near-optimal solutions are desired. However, the main point of this chapter is not to present a working algorithm, but to demonstrate that, under certain ideal conditions, the price communication technique alone can lead to the selection of an optimal basis.

For convenience in exposition, we summarize in the next section some properties of the Leontief systems which will be used throughout this report. In section 3 we describe the price communication decomposition procedure for two interconnected Leontief systems and prove some properties justifying its validity. The questions of convergence and finiteness of the method will be dealt with in section 4. Finally, in the last section we briefly indicate how to extend the procedure to the case of N Leontief systems.

2-2. Review of Some Properties of Leontief Systems.

We shall now, for future reference, briefly review some of the important properties of Leontief systems. Most of the general properties of Leontief matrices can be found in an expository paper by Woodbury [37], but we shall mostly refer to Dantzig and Wets [17] and Gale [21].

(1) Definition. An $(m \times n)$ matrix $(m > n)$ is called a Leontief matrix with substitution if and only if each column contains exactly one and each row at least one element which is positive.

It should be noted that a coefficient matrix of a linear program which is a Leontief matrix with substitution as defined by (1) can always be transformed by scaling the variables into a Leontief matrix whose positive elements are all equal to one, i. e., into a matrix of the form:

$$(2) \qquad A = \begin{bmatrix} 1 \cdots\cdots 1 & -a_{1,k+1} \cdots -a_{1e} & -a_{1n} \\ -a_{21} \quad -a_{2k} & 1 \qquad -a_{2e} & -a_{2n} \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\ -a_{m1} \quad -a_{mk} & -a_{m,k+1} \cdots 1 \cdots\cdots & 1 \end{bmatrix} \quad \text{all } a_{ij} \geq 0$$

Normally, it is this form which will be used in this report. In the same way we define a simple Leontief matrix.

(3) Definition. A matrix B is called a Leontief matrix if it is a square Leontief matrix with substitution, i. e., if each row has one and only one element which is positive.

If a Leontief matrix of order m has the property that the sum of the elements of every column is strictly positive, then the following theorem holds.

(4) Theorem. If A is a Leontief matrix of order m and satisfies the condition $e_m A > 0$, then the inverse of A exists and is a nonnegative matrix, where $e_m = (1, 1, \ldots, 1)$ or, more generally, $e_m = (w_1, w_2, \ldots, w_m)$ where $w_i > 0$.

It will suffice to show the theorem for $e_m = (1, 1, \ldots, 1)$ since the rows may be rescaled so that $w_i = 1$ and then the columns rescaled so that $a_{ii} = 1$ for all i. There exist several ways of proving this theorem. Usually the proof is based on the convergence of $I + \bar{A} + \bar{A}^2 + \ldots$ where $\bar{A}$ is a nonnegative matrix defined by $A = I - \bar{A}$ and $\bar{A}^n \to 0$ as $n \to \infty$ [21, p 301]. A different proof, based exclusively on the properties of the simplex method, hence algebraic, is given in [17]. ||

Considering now the linear program

(5) Minimize $Z = cx$ subject to $Ax = b$ and $x \geq 0$ where A is a Leontief matrix with substitution, we state the following well known results, detailed proofs of which can be found in [17].

(6) Lemma. If b is positive $(b > 0)$, then any feasible basis to problem (5) is a Leontief matrix.

Proof. It suffices to note that the feasibility requires at least one positive element in each row when $b > 0$. ||

Remark. A stronger form of the same lemma -- namely, that any basis to problem (5) to which corresponds a feasible nondegenerate solution is a Leontief matrix -- can be proved in a similar manner.

(7) Theorem. If $e_m A > 0$ then there exists a feasible solution to problem (5) for any $b \geq 0$.

Proof. If $e_m A > 0$, then any Leontief submatrix B of A has, according to (4), a nonnegative inverse. This implies that the basic set of activities defined by $w = B^{-1}b$ is feasible for any $b \geq 0$. ||

(8) Theorem. If B is a feasible basis to problem (5) for a positive b, then it is a feasible basis for any nonnegative b.

The proof of this theorem rests on the fact that if B is a feasible basis corresponding to $b > 0$, then B^{-1} is a nonnegative matrix; therefore, the same argument as above holds. The complete proof may be found in [17, p. 21]. ||

(9) Theorem. (Samuelson) If B is an optimal basis to problem (5) for a positive b, then B is an optimal basis for any nonnegative b.

Proof. The simplex multipliers π associated with B are independent of b and satisfy the dual constraints

(a) $\pi A \leq c$

because B is an optimal basis. But, according to (8), B is also a feasible basis for any $b \geq 0$; hence, the vector π is the same, the relation (a) still holds and the conclusion of the theorem follows. ||

(10) Remark. At this point it should be emphasized that theorem (9) is not equivalent to the statement that the optimal selection of alternative activities is independent of the right hand side b, as the following counter-example will show.

(11) Counter-example. Consider problem (5) with the data

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -.5 & 1 \end{bmatrix} \quad \text{and} \quad c = [1, 2, 4]$$

It can be easily checked that

(a) if $b = [1, 0]$ then $B' = \begin{bmatrix} 1 & 1 \\ 0 & -.5 \end{bmatrix}$ is an optimal basis

(b) if $b = [1, 1]$ then B' is not a feasible basis and the optimal basis is $B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) B^2 is also optimal when $b = [1, 0]$

This remark is not really restrictive when the coefficient matrix A satisfies the condition $e_m A > 0$, because we know, by (7), that there exists a feasible solution for any $b \geq 0$. Hence, in this case when an optimal solution for any $b \geq 0$ is desired, it suffices to apply the following rule.

(12) Rule. If problem (5) satisfies the condition $e_m A > 0$, then by solving it with an arbitrary $b > 0$, an optimal basis with the following two properties if found:

(i) it is a Leontief matrix (theorem 6)

(ii) it is also an optimal basis for any $b \geq 0$ (theorem 9).

2-3. Decomposition Procedure for Two Interconnected Leontief Systems.

We now present the decomposition procedure by price communication to solve a linear program of the form

Minimize Z subject to the constraints

$$x^1 \geq 0, \quad y^1 \geq 0, \quad y^2 \geq 0, \quad x^2 \geq 0$$

$$A^1x^1 + I_my^1 - I_my^2 \qquad = b^1$$

$$- I_my^1 + I_my^2 + A^2x^2 = b^2$$

$$c^1x^1 + d^1y^1 + d^2y^2 + c^2y^2 + c^2y^2 = Z \text{ (min)}$$

where A^1 and A^2 are Leontief matrices with substitution of size $(m \times n_1)$ and $(m \times n_2)$ respectively, and I_m is an identity matrix of order m.

We make the following assumptions which will be used in later developments:

(14) $\qquad e_mA^i > 0 \qquad$ for $i = 1, 2$

(15) $\qquad c^1 \geq 0, \quad c^2 \geq 0, \quad d^1 \geq 0, \quad d^2 \geq 0$

(16) $\qquad d^1 + d^2 > 0$

We shall relate the solution of problem (13) to the solutions of the following sequence of subproblems.

$$(17) \qquad a(t) \qquad \begin{cases} \text{Minimize } Z^1 \text{ satisfying} \\ x^1 \geq 0, \quad y^1 \geq 0 \\ A^1 x^1 + I_m y^1 = \tilde{b}^1 \\ c^1 x^1 + [d^1 + \pi^2(t-1)] y^1 = Z^1 \text{ (min)} \end{cases}$$

where $\tilde{b}^1$ is an arbitrary positive vector and $\pi^2(t-1)$ are the communicated optimal simplex multipliers from subproblem b(t-1) below.

$$(18) \qquad b(t) \qquad \begin{cases} \text{Minimize } Z^2 \text{ satisfying} \\ x^2 \geq 0, \quad y^2 \geq 0 \\ A^2 x^2 + I_m y^2 = \tilde{b}^2 \\ c^2 x^2 + [d^2 + \pi^1(t)] y^2 = Z^2 \text{ (min)} \end{cases}$$

where $\tilde{b}^2$ is an arbitrary positive vector and $\pi^1(t)$ are the communicated optimal prices from subproblem a(t).

All these problems are well defined except subproblem a(1) for which $\pi^2(0)$ is not specified. In fact, instead of a(1), we choose to solve the problem

$$\text{Min } Z^1 = c^1 x^1 \quad \text{subject to} \quad A^1 x^1 = b^1 \quad \text{and} \quad x^1 \geq 0$$

This can be achieved by setting $\pi^2(0) = \infty$ and solving a(1) as stated above.

We turn now to some of the properties of subproblems a(t) and b(t). Our first task is to show that their optimal simplex multipliers

$\pi^1(t)$ and $\pi^2(t)$ converge as t increases.

(19) <u>Lemma</u>. Under assumptions (14) and (15), $\pi^1(t) \geq 0$ and $\pi^2(t) \geq 0$.

<u>Proof</u>. Note that (14) guarantees that feasible solutions to a(t) and b(t) exist (see 7) and that, by assumption (15), $Z^1 \geq 0$ for problem a(1). Hence, by the duality theorem, an optimal solution to a(1) exists. Also, according to (12), we know that an optimal basis to a(1), say $B^1(1)$, is a Leontief matrix whose inverse is a nonnegative matrix. Consequently, since the basic cost vector $\gamma^1(1)$ is nonnegative, the simplex multipliers $\pi^1(1)$ are also nonnegative, i. e.,

$$\pi^1(1) = \gamma^1(1)[B^1(1)]^{-1} \geq 0$$

and, repeating the above argument for b(2), a(3), . . . , the conclusion follows. ||

(20) <u>Lemma</u>. $\pi^1(t)$ and $\pi^2(t)$ are monotonically nonincreasing sequences.

<u>Proof</u>. Because the set of activities of a(1) is a subset of the activities of a(2) it is obvious that

$$Z^1(1) \geq Z^1(2)$$

Furthermore, since B^1 is an optimal basis for <u>any</u> $b \geq 0$, we may choose $b = u_i$, a unit vector with 1 in i^{th} component; we then have by the duality theorem $Z^1(1) = \pi_i^1(1)$ and $Z^1(2) = \pi_i^1(2)$ and this for all $i \in M$. Hence,

$$\pi^1(1) \geq \pi^1(2).$$

But this implies that the cost vector of b(1) is greater than or equal to the cost vector of b(2); hence, $Z^2(1) \geq Z^2(2)$ and by the same argument as above

$$\pi^2(1) \geq \pi^2(2).$$

Similarly, assuming that $\pi^1(t-1) \geq \pi^1(t)$ we show that

$$\pi^2(t-1) \geq \pi^2(t)$$

thus completing the inductive proof of this lemma. ||

(21) Theorem. $\pi^1(t)$ and $\pi^2(t)$ converge respectively to π^1_* and π^2_* as $t \to \infty$.

Proof. By (19) and (20) both sequences are nonincreasing and bounded below by zero. So, they both converge. ||

We now turn to some definitions which will be used throughout this report.

(22) Definition. We call respectively importation set of the i^{th} block and production set of the i^{th} block the sets

$$I^i = \{j / j \in M \text{ and } y_j^i \text{ is a basic variable}\}$$

$$J^i = \{j / j \in N_i \text{ and } x_j^i \text{ is a basic variable}\}$$

Accordingly we denote the vector of basic <u>production variables</u> by

$$x_J = (x_j)_{j \in J}$$

and the vector of basic <u>importation variables</u> by

$$y_I = (y_j)_{j \in I}$$

Concerning these sets it will be useful to record that

(23) <u>Lemma</u>. Optimal solutions of two consecutive subproblems a(t) and b(t) have the property that

$$I^1 \cap I^2 = \varphi$$

Proof. By definition

$$\pi_j^1(t) = d_j^1 + \pi_j^2(t-1) \qquad \text{if} \quad j \in I^1$$

and

$$\pi_j^2(t) = d_j^2 + \pi_j^1(t) \qquad \text{if} \quad j \in I^2$$

Adding these relations we have

$$\pi_j^2(t) - \pi_j^2(t-1) = d_j^1 + d_j^2 > 0 \qquad \text{(by 16)}$$

which contradicts (20). This completes the proof. ||

Let us now examine how the solutions of subproblems a(t) and b(t) are related to the solution of the original problem. First we show that the matrix B(t) of this problem whose columns correspond to optimal bases $B^1(t)$ and $B^2(t)$ of a(t) and b(t) is a feasible basis for problem (13). It will be convenient to partition B(t) as follows:

$$(24) \qquad B(t) = \begin{bmatrix} B^1(t) & E^2(t) \\ E^1(t) & B^2(t) \end{bmatrix}$$

where we assume that the optimal bases $B^1(t)$ and $B^2(t)$ have their <u>positive elements on the diagonal</u> and the columns of $E^1(t)$ and $E^2(t)$ are defined by

$$(25) \qquad \begin{aligned} E_j^k(t) &= -u_j \quad \text{if} \quad j \in I^k, \quad \kappa = 1, 2 \\ &= 0 \text{ otherwise} \end{aligned}$$

<u>Remark</u>. It should be observed that the assumption that the positive elements of $B^1(t)$ and $B^2(t)$ are on the diagonals does not restrict the generality of this exposition since the simplex algorithm applied to Leontief systems preserves this characteristic; consequently, it suffices to put the original bases in the proper order.

(26) <u>Lemma</u>. <u>B(t) is a feasible basis for problem (13)</u>.

Proof. Let us suppose that B(t) is a nonsingular matrix and let us compute its inverse $\beta(t)$ by the partitioning method [26, p. 35].

Denoting by β^2 the nonnegative inverse of $B^2(t)$ (which, according to (4) and (6) exists) we have:

$$\beta(t) = \begin{bmatrix} \overline{\beta}^1 & \delta^2 \\ \delta^1 & \overline{\beta}^2 \end{bmatrix}$$

where

$$\overline{\beta}^1 = [B^1(t) - E^2(t)\,\beta^2 E^1(t)]^{-1}$$

$$\delta^2 = -\overline{\beta}^1 E^2(t)\,\beta^2$$

$$\delta^1 = -\beta^2 E^1(t)\,\overline{\beta}^1$$

$$\overline{\beta}^2 = \beta^2 - \beta^2 E^1(t)\,\delta^2$$

Under the assumption $I^1 \cap I^2 = \varphi$ we prove in chapter 3 that the matrix

$$\overline{B}^1 = B^1(t) - E^2(t)\,\beta^2 E^1(t)$$

is a Leontief matrix satisfying $e_m \overline{B}^1 > 0$. Hence, by (4) its inverse $\overline{B}^1$ exists and is a nonnegative matrix. As can be easily verified, this implies that the remaining matrix equations of (26) are all defined and are nonnegative. Hence, $\beta(t)$ is a nonnegative matrix and therefore $B(t)$ is a feasible basis for any nonnegative right hand side b. This completes the proof. ||

Thus far we have only seen that the simplex multipliers $\pi^1(t)$ and $\pi^2(t)$ converge and that our decomposition procedure gives us a sequence of feasible bases $B(t)$. To prove the validity of the whole method it remains to be shown that the sequence $\{B(t)\}$ will converge in a finite number of steps to an optimal basis of problem (13). This is done in the next section.

2-4. Convergence and Finiteness Proofs.

Let us turn now to the questions of convergence and finiteness of our decomposition procedure. We shall first note that, if for any two consecutive iterations the simplex multipliers are equal, then the iterative process is ended. More precisely

(28) Lemma. If $\pi^1(t-1) = \pi^1(t)$ or $\pi^2(t-1) = \pi^2(t)$ then the basis $B(t)$ is an optimal basis for problem (13).

Proof. Pricing out the activities of problem (13) with the simplex multipliers $\pi^1(t)$ and $\pi^2(t)$ we have

$$
\begin{aligned}
c^1 - \pi^1(t)A^1 &\geq 0 \\
c^2 - \pi^2(t)A^2 &\geq 0 \qquad (29)\\
d^2 + \pi^1(t) - \pi^2(t) &\geq 0
\end{aligned}
$$

as a direct consequence of the definitions of $\pi^1(t)$ and $\pi^2(t)$. However, nothing can be said about the sign of the expression

$$d^1 + \pi^2(t) - \pi^1(t) \qquad (30)$$

since the equivalent columns in the problem $a(t)$ price out

$$d^1 + \pi^2(t-1) - \pi^1(t) \geq 0 \qquad (31)$$

Nevertheless, when $\pi^2(t) = \pi^2(t-1)$ then (30) is identical to (31). Hence, noting that $\pi^1(t-1) = \pi^1(t)$ implies that $\pi^2(t-1) = \pi^2(t)$, the conclusion of the lemma follows. ||

(32) Theorem. The limiting simplex multipliers π_*^1 and π_*^2 are optimal simplex multipliers to problem (13).

Proof. It suffices to note that in the proof of the preceding lemma

(i) the relations (29) must hold in the limit also;

(ii) the expression (30) is asymptotically greater than or equal to zero since $\pi^2(t-1) - \pi^2(t) \to 0$ as $t \to \infty$. ||

So far we can conclude that, since $\pi^1(t)$ and $\pi^2(t)$ converge to π_*^1 and π_*^2, the decomposition procedure by price communication tends to give an optimal feasible basis for problem (13). In fact we shall show now that the choice of this optimal basis is reached in a finite number of iterations.

(33) Theorem. An optimal basis $B(t)$ to problem (13) is reached in a finite number of iterations.

Proof. Let $[\bar{x}^1(t), \bar{y}^1(t), \bar{y}^2(t), \bar{x}^2(t)]$ be the basic feasible solution to problem (13) which is associated with the basis $B(t)$. By multiplying the rows of the coefficient matrix (13) by $\pi^1(t)$ and $\pi^2(t)$ and subtracting them from the cost form, it can be easily verified that the objective function $Z(t)$ of (13) has the value

$$Z(t) = \pi^1(t)b^1 + \pi^2(t)b^2 + [\pi^2(t) - \pi^2(t-1)]\,\bar{y}^1(t).$$

According to (20), $\pi^2(t) - \pi^2(t-1) \leq 0$; therefore

$$Z(t) \leq \pi^1(t)b^1 + \pi^2(t)b^2.$$

Furthermore, also by (20), we know that there exists for a given $\alpha > 0$ an integer N such that

$$\left.\begin{array}{l} \pi^1(t)b^1 < \pi^1_* b^1 + \alpha/2 \\ \pi^2(t)b^2 < \pi^2_* b^2 + \alpha/2 \end{array}\right\} \quad \text{for all } t > N$$

Hence, by theorem (32) we have

$$Z(t) < Z\ \min + \alpha \qquad \text{for all } t > N$$

Now, let us suppose that B^o is a nonoptional basis of problem (13). The corresponding value of the objective function, say Z^o, can be written as

$$Z^o = Z\ \min + h \qquad (h > 0)$$

Therefore, if $\alpha < h$, we have

$$Z(t) < Z^o \qquad \text{for all} \quad t > N$$

This means that, if t is sufficiently large, the nonoptional basis B^o cannot be chosen by our procedure. This holds for all nonoptimal bases and therefore the conclusion of this theorem follows. ||

2-5. Extension to N Interconnected Leontief Systems.

The preceding analysis can be readily extended to the case of N interconnected Leontief systems because it can be easily shown that all the results of the previous analysis hold in this case also. To illustrate the procedure applied to such a case we consider the following linear program stated in detached coefficient form:

Minimize Z subject to

(34)

x^1	x^2	x^3	y^{12}	y^{13}	y^{21}	y^{23}	y^{31}	y^{32}	Constants
A^1			I	I	$-I$		$-I$	$-I$	$b^1 \geq 0$
	A^2		$-I$		I	I			$b^2 \geq 0$
		A^3		$-I$		$-I$	I	I	$b^3 \geq 0$
c^1	c^2	c^3	d^{12}	d^{13}	d^{21}	d^{23}	d^{31}	d^{32}	Z (min)

Where all identity matrices I are of order m.

In this case, the subproblems to be solved are

(35)

$$a(t) \begin{cases} \text{Minimize } Z^1 \text{ satisfying} \\ x^1 \geq 0, \quad y^{12} \geq 0, \quad y^{13} \geq 0 \\ A^1 x^1 + I_m y^{12} + I_m y^{13} = \tilde{b}^1 \\ c^1 x^1 + [d^{12} + \pi^2(t-1)] y^{12} + [d^{13} + \pi^3(t-1)] y^{13} = Z^1 (\text{min}) \end{cases}$$

$$b(t) \begin{cases} \text{Minimize } Z^2 \text{ satisfying} \\ x^2 \geq 0, \quad y^{21} \geq 0, \quad y^{23} \geq 0 \\ A^2 x^2 + I_m y^{21} + I_m y^{23} = \tilde{b}^2 \\ c^2 x^2 + [d^{21} + \pi^1(t)] y^{21} + [d^{23} + \pi^3(t-1)] y^{23} = Z^2 (\text{min}) \end{cases}$$

$$c(t) \begin{cases} \text{Minimize } Z^3 \text{ satisfying} \\ x^3 \geq 0, \quad y^{31} \geq 0, \quad y^{32} \geq 0 \\ A^3 x^3 + I_m y^{31} + I_m y^{32} = \tilde{b}^3 \\ c^3 x^3 + [d^{31} + \pi^1(t)] y^{31} + [d^{32} + \pi^2(t)] y^{32} = Z^3 (\text{min}) \end{cases}$$

where $\tilde{b}^1$, $\tilde{b}^2$, and $\tilde{b}^3$ are arbitrary positive vectors, and $\pi^1(t)$, $\pi^2(t)$ and $\pi^3(t)$ are optimal simplex multipliers to the subproblems a(t), b(t) and c(t) respectively.

It should be noted that the above subproblems can be further simplified by introducing importation cost vectors $d^k(t)$ defined, for instance for $k = 1$, by

$$d_j^1(t) = \min\left(d_j^{12} + \pi_j^2(t-1),\ d_j^{13} + \pi_j^3(t-1) \right) \quad \text{for all} \quad j \in M$$

With this simplification subproblem a(t) becomes

$$(36) \qquad a^1(t) \quad \left\{ \begin{array}{l} \text{Minimize } Z^1 \text{ subject to} \\ x^1 \geq 0, \quad y^1 \geq 0 \\ A^1x^1 + Iy^1 = \tilde{b}^1 \\ c^1x^1 + d^1y^1 = Z^1 \text{ (min)} \end{array} \right.$$

Thus, the subproblems have the same form as those described in the preceding sections, with the exception that a kind of indicator vector has to be set up to determine the origin of every importation vector.

CHAPTER III

DECOMPOSITION OF TWO INTERCONNECTED LEONTIEF SYSTEMS BY SQUARE BLOCK TRIANGULARIZATION

3-1. Introduction.

From now on this report will be essentially devoted to the development of a decomposition procedure which will be called decomposition by square block triangularization. The underlying idea of this method is to transform successively the linear program to be solved into a series of equivalent linear programs which have the property that their current feasible bases are square block triangular, i. e., have the form

$$\begin{array}{|c|c|c|} \hline B^1 & E^1 & E^2 \\ \hline 0 & B^2 & E^3 \\ \hline 0 & 0 & B^3 \\ \hline \end{array}$$

where B^1, B^2 and B^3 are square matrices. Under these conditions, we shall see that improved solutions to these equivalent problems can be easily found by solving smaller subproblems which will be referred to as improvement subprograms. Thus, we shall see that, starting with a feasible basis, we can, through a sequence of improvements, solve the original problem by solving a series of smaller linear programs. We mention immediately that in practice the equivalent problems are never set up; they serve only for convenience of exposition. As far as the improvement subprograms are concerned, their coefficient matrix is not constant, and, therefore, they will have to be set up at each iteration. However, it turns out that for the case of interconnected systems they are easily obtained by a slight modification of the exchange matrices.

In this chapter we shall start the study of the decomposition method by square block triangularization by developing it in connection with the problem studied in the preceding chapter. There are two main reasons for doing this:

(i) we finish the study started in the preceding chapter;

(ii) the method has remarkable properties in this case which will simplify the exposition and illustrate once more the ideal behavior of Leontief systems. This illustration is important because it will show very clearly what is the minimum exchange of information which will be required to solve efficiently by decomposition a linear program based on interconnected systems.

Consequently, we shall again be essentially concerned with linear programming problems of the form

$$(1)\quad \begin{cases} \text{Minimize} & Z = c^1x^1 + d^1y^1 + d^2y^2 + c^2x^2 \\ \text{subject to} & x^1 \geq 0, \quad y^1 \geq 0, \quad y^2 \geq 0, \quad x^2 \geq 0 \\ & A^1x^1 + I_my^1 - I_my^2 = b^1 \\ & -I_my^1 + I_my^2 + A^2x^2 = b^2 \end{cases}$$

where A^1 and A^2 are Leontief matrices with substitution, and, as before, we shall make the following assumptions.

(2) $e_mA^1 > 0$ and $e_mA^2 > 0$

(3) $c^1 \geq 0, \quad c^2 \geq 0, \quad d^1 \geq 0, \quad d^2 \geq 0$

(4) $d^1 + d^2 > 0$

Remarks. (i) It should be noted that assumption (4) is a necessary condition for the existence of an optimal solution to problem (1), whereas the assumptions (2) and (3) are only sufficient conditions made here for the convenience of exposition.*

(ii) We recall (2-12) that problem (1) can best be solved by taking arbitrary positive right hand sides b^1 and b^2 because the optimal basis obtained with these values is optimal for any $b^1 \geq 0$ and any $b^2 \geq 0$. Accordingly, without loss of generality, we shall assume throughout this chapter that $b^1 > 0$ and $b^2 > 0$ (except in the statement of the algorithm where we replace b^1 and b^2 by e_m for precisely the reason indicated above).

---

*In fact, it can be shown that (3) is superfluous.

In this chapter we first give some additional properties of Leontief matrices in relation to square block triangularity in section 2. In section 3 we list, for future reference, a series of simple results related to the transformations of the original problem into equivalent problems; also we discuss the computational aspects of such transformations. In section 4 the validity of the decomposition method by square block triangularization to solve problem (1) will be proved, and in section 5 the corresponding algorithm will be stated. Finally, in section 6 we briefly compare this method with that of the preceding chapter as well as with the decomposition method of Dantzig and Wolfe.

3-2. Square Block Triangularity and Leontief Matrices.

In this section we prove some further properties of Leontief matrices which will be needed for the decomposition procedure described in this chapter. However, since these properties have an interest of their own, we shall treat this section rather independently, referring only to the theorems of section (2-2), duplicating a few minor results of the preceding chapter.

We consider a square Leontief matrix of order m partitioned as

$$A = \begin{bmatrix} A^1 & A^3 \\ A^2 & A^4 \end{bmatrix} \tag{5}$$

where A^1 and A^4 are square Leontief matrices of order p and q

respectively and, consequently, A^2 and A^3 are nonpositive matrices.

Let us turn our attention first to the steps which are necessary to transform the above matrix A into a square block triangular matrix $\overline{A}$.

(6) Definition. The matrix $\overline{A}$ is said to be block triangular if it has the form

$$\overline{A} = PA = \begin{bmatrix} \overline{A}^1 & \overline{A}^2 \\ 0 & \overline{A}^4 \end{bmatrix}$$

and it is said to be square block triangular if $\overline{A}^1$ and $\overline{A}^4$ are square matrices.

Assuming that submatrix A^1 is nonsingular and denoting its inverse by α' we define a nonsingular transformation matrix P by

(7) $$P = \begin{bmatrix} I_p & 0 \\ -A^2\alpha' & I_q \end{bmatrix}$$

Premultiplying A by P we obtain the square block triangular matrix

(8) $$\overline{A} = \begin{bmatrix} A^1 & A^3 \\ 0 & \overline{A}^4 \end{bmatrix}$$

where

(9) $$\overline{A}^4 = A^4 - A^2\alpha^1A^3$$

We shall see now that this transformation preserves the Leontief properties of matrix A^4. More precisely

(10) Theorem. If the matrix A has the property that $e_m A > 0$, then $\bar{A}^4$ is a Leontief matrix having also the property that $e_q \bar{A}^4 > 0$.

Proof. The assumption $e_m A > 0$ can also be written as

$$\text{(11)} \qquad \begin{aligned} &\text{(i)} \quad e_p A^1 + e_q A^2 > 0 \\ &\text{(ii)} \quad e_p A^3 + e_q A^4 > 0 \end{aligned}$$

Noting that $e_p A^1 > 0$ we know by theorem (2-4) that the inverse α^1 of A^1 exists and is a nonnegative matrix. Hence, the transformation matrix P as well as the matrix $\bar{A}$ exist and $\bar{A}^4$ is well defined. By definition of $\bar{A}^4$ we have

$$e_q \bar{A}^4 = e_q A^4 - e_q A^2 \alpha^1 A^3$$

But, by assumption (ii), $e_q A^4 > - e_p A^3$; therefore,

$$\text{(12)} \qquad e_q \bar{A}^4 > - (e_p + e_p A^2 \alpha^1) A^3$$

By definition A^3 is a nonpositive matrix; therefore, in order to prove that

$$\text{(iii)} \quad e_q \bar{A}^4 > 0$$

holds, it suffices to verify that

$$(13) \qquad e_p + e_p A^2 \alpha^1 = (e_p A^1 + e_p A^2)\alpha^1$$

is nonnegative. This is so by assumption (i) and by the fact that α^1 is a nonnegative matrix. Hence (iii) is satisfied.

It remains to be proved that $\overline{A}^4$ is a Leontief matrix. We note that (iii) implies that each column has at least one positive element. Furthermore, $A^2 \alpha^1 A^3$ is a nonnegative matrix; therefore, the positive elements of

$$\overline{A}^4 = A^4 - A^2 \alpha^1 A^3$$

can only be those of A^4. Hence $\overline{A}^4$ is a Leontief matrix. ||

We shall extend the preceding result to the case of a square non-singular Leontief matrix of order $2m$ having the form

$$(14) \qquad B = \left[\begin{array}{c|c|c|c} D^1 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & D^2 \end{array}\right] = \begin{bmatrix} B^1 & E^2 \\ E^1 & B^2 \end{bmatrix}$$

where D^1 and D^2 are Leontief matrices, of size $(m \times p)$ and $(m \times q)$ respectively, satisfying the conditions

$$(15) \qquad e_m D^1 > 0 \qquad \text{and} \qquad e_m D^2 > 0$$

Also, the positive elements of B are assumed to be on the diagonal.

However, before proving our next theorem, let us pause to introduce some definitions and to state a necessary and sufficient condition for a matrix B to be nonsingular.

(16) Definitions. We call B a Leontief matrix with interconnections, and the columns B_j of B which have an element -1 interconnection columns.

The column of B^1 corresponding to an interconnection column B_j of B is called an importation column, whereas the column of E^1 corresponding to the same B_j is called exportation column.

Finally, we call importation set the set I^i defined by

$$I^i = \{ j \mid j \in M \text{ and } B^i_j \text{ is an importation column} \} \text{ for } i = 1, 2$$

(17) Lemma. A Leontief matrix with interconnections is nonsingular if and only if $I^1 \cap I^2 = \Phi$.

Proof. Consider the matrix $\tilde{B}$ obtained by multiplying the rows of B having an element -1 by the scalar $(1 - \epsilon) > 0$, $\epsilon > 0$, i.e.,

$$\tilde{B} = MB = \begin{bmatrix} M^1 & 0 \\ 0 & M^2 \end{bmatrix} \times \begin{bmatrix} B^1 & E^2 \\ E^1 & B^2 \end{bmatrix}$$

where M^1 and M^2 have the form

$$M = \begin{bmatrix} 1-\epsilon & & & & \\ & 1-\epsilon & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

To prove that B is nonsingular it suffices to prove that there exists an $1 > \epsilon > 0$ for which $\tilde{B}$ is nonsingular. To show this we observe that, as a consequence of (15), there must exist an $\epsilon > 0$ which is small enough to preserve property (15) for $\tilde{B}$ also. Furthermore, if $I^1 \cap I^2 = \varphi$, then the columns $\tilde{B}_j$ of $\tilde{B}$ corresponding to interconnection columns have also the property $e_{2m}B_j > 0$; hence, according to theorem (2-4), $\tilde{B}$ is nonsingular. Finally to prove necessity we note that if $j \in I^1$, and if $j \in I^2$, then $B_j = -B_{j+m}$ and B is singular, contradicting assumption. ||

(18) <u>Corollary 1.</u> <u>If $I^1 \cap I^2 = \varphi$, then the inverse of B is a nonnegative matrix.</u>

<u>Proof.</u> From the preceding proof we know that

$$B^{-1} = \tilde{B}^{-1}M$$

where $\tilde{B}^{-1}$ is nonnegative by theorem (2-4) and M is nonnegative by definition. ||

(19) <u>Corollary 2.</u> <u>If B is a basis of a linear program, then B is a feasible basis for any nonnegative right hand side b.</u>

<u>Proof.</u> $Bw = b \Rightarrow w = B^{-1}b \geq 0$ by corollary 1. ||

Turning now to the triangularization of B we premultiply it by the nonsingular transformation matrix

(20) $$P = \begin{bmatrix} I_m & 0 \\ -E^1\beta^1 & I_m \end{bmatrix},$$

and we obtain the square block triangular form

(21) $$\overline{B} = PB = \begin{bmatrix} B^1 & E^2 \\ 0 & \overline{B}^2 \end{bmatrix}$$

where

(22) $$\overline{B}^2 = B^2 - E^1\beta^1E^2$$

(23) Theorem. $\overline{B}^2$ is a Leontief matrix having the property that

$$e_m\overline{B}^2 > 0$$

Proof. Note that, since $e_{2m}B > 0$, the proof of theorem (10) holds when we replace the signs $>$ by $\geq$; however, we want to prove that $e_m\overline{B}^2 > 0$.

Considering (12), it can be easily verified that for any $j \notin I^2$ strict inequality still holds and therefore

$$(e_m\overline{B}^2)_j > 0 \qquad \text{for all} \quad j \notin I^2$$

Hence, it remains only to prove that the inequality holds when $j \in I^2$. To do so, we first observe that, in the present situation, (13) becomes

$$(e_m B^1 + e_m E^1)\beta^1$$

where, by assumption,

$$(e_m B^1 + e_m E^1)_j > 0 \quad \text{if } j \notin I^1$$
$$= 0 \quad \text{if } j \in I^1$$

Furthermore, the inverse β^1 of B^1 must have the form

$$\beta^1 = \begin{bmatrix} H & 0 \\ G & I \end{bmatrix}$$

where H is a nonsingular matrix with only nonnegative elements. Therefore,

$$\left[(e_m B^1 + e_m E^1)\beta^1\right]_j > 0 \quad \text{if } j \notin I^1$$

But, by assumption, $I^1 \cap I^2 = \varphi$; therefore, for $j \in I^2$

$$(e_m \overline{B}^2)_j = -\left[(e_m B^1 + e_m E^1)\beta^1 E^2\right]_j > 0$$

since E_j^2 is a unit vector with unit in jth component and $j \notin I^1$ for $j \in I^2$. This completes the proof. ||

(24) Corollary. If $\theta^1 = -E^1\beta^1$, then $0 \leq (e_m\theta^1)_j < 1$ for all $j \notin I^1$.

Proof. By (17) and (18) there exists a matrix

$$B = \begin{bmatrix} B^1 & E^2 \\ E^1 & B^2 \end{bmatrix}$$

such that $I^1 \cap I^2 = \varphi$ and $I^1 \cup I^2 = M$ which is a Leontief matrix with interconnections. Therefore, according to theorem (23)

$$e_m\overline{B}^2 = e_m B^2 + e_m \theta^1 E^2 > 0 .$$

If $j \in I^2$, then the j^{th} component of the above vector can be written as

$$1 - e_m \theta^1_j > 0 .$$

Noting that $\theta^1_j \geq 0$, the conclusion of the corollary follows. ||

Several remarks are in order now. First, it will be convenient to introduce the notations

$$\theta^1 = -E^1\beta^1 \qquad \text{and} \qquad \theta^2 = -E^2\beta^2$$

whereby (21) and (22) can be written as

$$\text{(25)} \qquad \overline{B} = PB = \begin{bmatrix} I_m & 0 \\ \theta^1 & I_m \end{bmatrix} \times \begin{bmatrix} B^1 & E^2 \\ E^1 & B^2 \end{bmatrix} = \begin{bmatrix} B^1 & E^2 \\ 0 & \overline{B}^2 \end{bmatrix}$$

and

$$\overline{B}^2 = B^2 + \theta^1 E^2 \tag{26}$$

It should be noted that θ^1 and $\theta^1 E^2$ are submatrices of β^1; hence, once β^1 is known, it is very easy to transform B as defined by (14) into $\overline{B}$. We illustrate this point by showing below the structures of the various matrices involved.

$$B^1 = \left[\begin{array}{c|cc} D^1 & 0 & \\ \hline & 1 & \\ & & 1 \end{array}\right] \qquad \beta^1 = \left[\begin{array}{ccc|cc} x & x & x & & \\ x & x & x & 0 & \\ x & x & x & & \\ \hline * & * & x & 1 & \\ * & * & x & & 1 \end{array}\right] \qquad \theta^1 = \left[\begin{array}{ccc|cc} & 0 & & 0 & \\ \hline * & * & x & 1 & \\ * & * & x & & 1 \end{array}\right]$$

$$B^2 = \left[\begin{array}{cc|c} 1 & & \\ & 1 & D^2 \\ & & \end{array}\right] \qquad \overline{B}^2 = \left[\begin{array}{cc|c} 1 & & \\ & 1 & D^2 \\ \hline -*- & * & \\ -*- & * & \end{array}\right]$$

where $*$ indicates an element β^1_{ij} such that $i \in I^1$ and $j \in I^2$.

An economic interpretation of the elements β^1_{ij} modifying B^2 can be given if one considers the matrix B as a feasible cooperation plan between two Leontief economies I and II for which the basic activity levels are defined by

$$B\,[w^1, w^2] = \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$$

where w^1_j for $j \in I^1$ and w^2_k for $k \in I^2$ represent importation activities of economy I and economy II respectively. Recalling that $I^1 \quad I^2 = \emptyset$ it can be easily seen that $\left(\frac{dw^1}{db^1_j}\right)_i = \beta_{ij}$; hence,

(27) Interpretation. If $i \in I^1$ and $j \in I^2$ then B_{ij} represents the additional amount of commodity i that economy I has to import when economy II increases its importation of commodity j by one unit.

Accordingly, we introduce the following

(28) Definition. We shall call the matrix θ^1 an adjustment matrix and the elements β^1_{ij} for which $i \in I^1$ and $j \in I^2$ adjustment coefficients.

3-3. Equivalent Problems.

In this section we point out the advantages of transforming problem (1) into an equivalent linear programming problem having a square block triangular basis, and derive, for future reference, a series of simple equations related to this transformation. Further, we discuss some of the computational aspects of the square block triangularization method.

Assuming that we know a feasible basis B of problem (1), we consider its two transforms $\overline{B}$ and $\tilde{B}$, defined by (21) and (29) respectively. We recall that these matrices have the form

(29) $$B = \begin{bmatrix} B^1 & E^2 \\ E^1 & B^2 \end{bmatrix} \qquad \overline{B} = \begin{bmatrix} B^1 & E^2 \\ 0 & \overline{B}^2 \end{bmatrix} \text{ and } \tilde{B} = \begin{bmatrix} \tilde{B}^1 & 0 \\ E^1 & B^2 \end{bmatrix}$$

where

$$\overline{B}^2 = B^2 + \theta^1 E^2 \qquad \theta^1 = -E^1\beta^1$$

$$\tilde{B}^1 = B^1 + \theta^2 E^1 \qquad \theta^2 = -E^2\beta^2$$

Next, since the matrix P^1 which transforms B into $\overline{B}$ is a nonsingular matrix, we can transform problem (1) into an equivalent problem by premultiplying the coefficient matrix of the former by P^1. This new problem, which will be referred to as the <u>equivalent problem corresponding to the basis B</u>, can be stated as

$$(20)\quad \begin{cases} \text{Minimize } Z = c^1x^1 + d^1y^1 + d^2y^2 + c^2x^2 \\ \text{subject to} \quad A^1x^1 + I_m y^1 - I_m y^2 = b^1 \\ \overline{A}^1x^1 - \overline{T}^2y^1 + \overline{T}^2y^2 + A^2x^2 = b^2 \\ x^1 \geq 0, \quad y^1 \geq 0, \quad y^2 \geq 0, \quad x^2 \geq 0 \end{cases}$$

where

$$\overline{A}^1 = \theta^1 A^1$$

$$\overline{T}^2 = I_m - \theta^1$$

$$\overline{b}^2 = b^2 + \theta^1 b^1$$

We note that, by definition, we can solve either the original problem or the equivalent problem since both have the same solution set. Of course, for reasons which will become apparent in the next section, we will choose to solve the latter. However, before we go into further details, it will be useful to establish the following relations between these two problems.

Consider first the problem of the determination of the values of the simplex multipliers $[\pi^1\pi^2]$ and $[\bar{\pi}^1\bar{\pi}^2]$ associated with B and $\overline{B}$ respectively. Denoting the cost vector corresponding to these bases by $[\gamma^1, \gamma^2]$ we have, by definition

$$
(31) \qquad \begin{cases} (\pi^1, \pi^2)\,B = (\gamma^1, \gamma^2) \quad \text{and} \\ (\bar{\pi}^1, \bar{\pi}^2)\,\overline{B} = (\gamma^1\gamma^2) \end{cases}
$$

Accordingly, it can be easily verified that

$$
(32a) \qquad \begin{cases} \bar{\pi}^1 = \gamma^1\beta^1 \\ \bar{\pi}^2 = (\gamma^2 - \bar{\pi}^1 E^2)\bar{\beta}^2 \end{cases}
$$

Hence, recalling that by definition $B^{-1} = \overline{B}^{-1}P$, we have

$$
(\pi^1, \pi^2) = (\bar{\pi}^1, \bar{\pi}^2)\,P,
$$

i.e.,

$$
(33a) \qquad \begin{cases} \pi^1 = \bar{\pi}^1 + \bar{\pi}^2\theta^1 \\ \pi^2 = \bar{\pi}^2 \end{cases}
$$

Similarly, if $(\tilde{\pi}^1, \tilde{\pi}^2)$ are the simplex multipliers associated with $\tilde{B}$ we have

$$
(32b) \qquad \begin{cases} \tilde{\pi}^1 = (\gamma^1 - \tilde{\pi}^2 E^1)\tilde{\beta}^1 \\ \tilde{\pi}^2 = \gamma^2\beta^2 \end{cases}
$$

and

$$
(33b) \qquad \begin{cases} \pi^1 = \tilde{\pi}^1 \\ \pi^2 = \tilde{\pi}^2 + \tilde{\pi}^1\theta^2 \end{cases}
$$

Finally, the following relations will also be useful

$$(34a)\qquad \begin{cases} \tilde{\pi}^1 = \bar{\pi}^1 + \bar{\pi}^2\theta^1 \\ \tilde{\pi}^2 = \bar{\pi}^2 - \tilde{\pi}^1\theta^2 \end{cases}$$

or

$$(34b)\qquad \begin{cases} \bar{\pi}^1 = \tilde{\pi}^1 - \bar{\pi}^2\theta^1 \\ \bar{\pi}^2 = \tilde{\pi}^2 + \tilde{\pi}^1\theta^2 \end{cases}$$

Two remarks must be made now. First, note that, if the column eliminated during a simplex step applied to problem (30) belongs to the matrix $\overline{B}^2$, then the new prices $\bar{\pi}$ can be obtained by computing only the inverse of the new basis $\overline{B}^2$ since $\bar{\pi}^1$ remains unchanged. This remark will be important in the following section.

Next,the above relations hold whether or not D^1 and D^2 are Leontief matrices; the only conditioned required is $I^1 \cap I^2 = \varphi$.

The next problem we consider is the representation, say $A_s^{\#}$, of a vector A_s in terms of the basis B. By definition we have

$$(35)\qquad A_s^{\#} = B^{-1}A_s = \overline{B}^{-1}\overline{A}$$

where $\overline{A}_s = P^1A_s$ is equal to

$$(36)\qquad \begin{aligned} \overline{A}_s^1 &= A_s^1 \\ \overline{A}_s^2 &= A_s^2 + \theta^1 A_s^1 \end{aligned}$$

Consequently, we have

$$(37) \qquad \begin{aligned} (A_S^1)^{\#} &= \beta^1[A_s^1 - E^2(A_s^2)^{\#}] \\ (A_s^2)^{\#} &= \overline{\beta}^2[A_s^2 + \theta^1 A_s^1] \end{aligned}$$

Obviously, the above formula can be applied to compute the feasible solution corresponding to the basis B; it suffices to replace A_s by $b = (b^1, b^2)$ to obtain

$$(38) \qquad \begin{aligned} w^1 &= \beta^1[b^1 - E^2 w^2] \\ w^2 &= \overline{\beta}^2[b^2 + \theta^1 b^1] \end{aligned}$$

Finally, we have to indicate the equations relating the inverses $\beta^1, \beta^2, \overline{\beta}^1$ and $\overline{\beta}^2$ of the matrices $B^1, B^2, \overline{B}^1$ and $\overline{B}^2$ respectively. However, before doing so we shall introduce a very convenient formula due to Sherman and Morrison [35] which can best be stated [3, p. 36] as follows:

(39) Lemma. If B is a nonsingular matrix of order n and if

$$\overline{B} = B + CAD$$

where A is a nonsingular matrix of order m and C and D are matrices of size $(n \times m)$ and $(m \times n)$ respectively, then, provided that M^{-1} exists,

$$\overline{B}^{-1} = B^{-1} - B^{-1}CM^{-1}DB^{-1}$$

where

$$M = [A^{-1} + DB^{-1}C]$$

Proof. It is easy to check that $\overline{B}\,\overline{B}^{-1} = I$. This completes the proof. ||

A particular case of this lemma is the following well known result [9, p. 198]:

(40) Corollary. If a column P_r of a nonsingular matrix B is replaced by a column P_s, i.e., if

$$\overline{B} = B + (P_s - P_r)u_r'$$

then, provided that $\overline{P}_{rs} \neq 0$,

$$\overline{B}^{-1} = B^{-1} - \frac{1}{\overline{P}_{rs}}(\overline{P}_s - u_r)B_r^{-1}$$

where $\overline{P}_s = B^{-1}P_s$ and B_r^{-1} is the r^{th} row of B^{-1}.

Proof. Applying (39) we have

$$\overline{B}^{-1} = B^{-1} - \frac{B^{-1}(P_s - P_r)B_r^{-1}}{1 + B_r^{-1}(P_s - P_r)}$$

which is equivalent to the form given above, since by definition $\overline{P}_s = B^{-1}P_s$ and $B^{-1}P_r = u_r$. ||

Remark. Note that $\overline{P}_{rs} \neq 0$ is a necessary and sufficient condition for $\overline{B}$ to be nonsingular.

We are now in a position to give the relations between $\beta^1, \beta^2, \overline{\beta}^1$ and $\overline{\beta}^2$. To do this we extend our notations

$$\theta^1 = -E^1\beta^1 \quad \text{and} \quad \theta^2 = -E^2\beta^2$$

to

$$\overline{\theta}^1 = -E^1\overline{\beta}^1 \quad \text{and} \quad \overline{\theta}^2 = -E^2\overline{\beta}^2$$

We have three cases to consider.

(i) When β^1 and β^2 are known, then

$$\overline{B}^2 = B^2 - E^1\beta^1E^2 = (I - \theta^1\theta^2)\,B^2;$$

hence

$$\overline{\beta}^2 = \beta^2[I - \theta^1\theta^2]^{-1} \tag{41}$$

and, similarly,

$$\overline{\beta}^1 = \beta^1\,[I - \theta^2\theta^1]^{-1}$$

(ii) When β^1 and $\overline{\beta}^2$ are known, then

$$B^2 = \overline{B}^2 + E^1\beta^1E^2 = (I + \theta^1\overline{\theta}^2)\,\overline{B}^2;$$

so

$$\beta^2 = \overline{\beta}^2\,[I + \theta^1\overline{\theta}^2]^{-1} \tag{42}$$

and, similarly, when β^2 and $\tilde{\beta}^1$ are known

$$\beta^1 = \tilde{\beta}^1\,[I + \theta^2\tilde{\theta}^1]^{-1}$$

(iii) When $\tilde{\beta}^1$ and β^2 are known, one can compute directly $\overline{\beta}^2$ by applying (39); the inverse of $\overline{B}^2 = B^2 - E^1\beta^1E^2$ is

$$\overline{\beta}^2 = \beta^2 + \beta^2E^1M^{-1}E^2\beta^2\,.$$

where

$$M = [\,B^1 - E^2\beta^2E^1] = \overline{B}^1;$$

hence

$$\overline{\beta}^2 = \beta^2\,[I + \overline{\theta}^1\theta^2] \tag{43}$$

and, similarly,

$$\tilde{\beta}^1 = \beta^1 [I + \bar{\theta}^2 \theta^1]$$

3-4. Decomposition Procedure for Two Interconnected Leontief Systems.

We come now to the object of this chapter which is the development of an efficient decomposition procedure to solve problem (1). As was done in Chapter II, we shall set up two subproblems which will be based on the matrices A^1 and A^2 respectively, and, as before, these subproblems will be perfectly symmetric in the sense that the required kind of information exchanged is the same in both directions. The difference between both methods lies essentially in the fact that here the simplex multipliers of the subproblems will also be the simplex multipliers of the equivalent problem. To achieve this we will have to adjust the coefficient matrices of the subproblems at each iteration. This can be done quite easily, as we shall presently see.

First, recall that we assumed that b^1 and b^2 are positive; therefore, according to (2-6), any feasible basis to problem (1) must be a Leontief matrix. Consequently, if the simplex method is applied to solve problem (1), then the pivot row will be determined by the position of the positive element in the column to be introduced. Thus,

(44) Remark. In any feasible basis B to problem (1) the submatrices B^1 and B^2 correspond to coefficient columns associated with (x^1, y^1) and (x^2, y^2) respectively.

Next, we assume that we have a feasible basis, say B, to problem (1), and we consider the equivalent problem (30) associated with it. In order to improve the basic solution corresponding to B we apply the simplex method to this problem, limiting the pivot choice to the variables x^2 and y^2 only. It is a simple matter now to show that the simplex steps thus performed can be carried out simply by solving the following subproblem which will be referred to as the improvement subprogram based on B^2.

(45)
$$\left\{\begin{array}{l} \text{Minimize } Z^2 \text{ satisfying} \\ A^2x^2 + T^2y^2 = b^2 \\ c^2x^2 + t^2y^2 = Z^2 \ (\min) \\ x^2 \geq 0, \quad y^2 \geq 0 \end{array}\right.$$

where

(46)
$$T^2 = [I_m - \theta^1]_{\bar{I}^1} \qquad \text{and} \qquad \bar{I}^1 = M - I^1$$
$$t^2 = [d^2 + \bar{\pi}^1]_{\bar{I}^1} \qquad \text{where } \bar{\pi}^1 \text{ is defined by (32a)}$$

(47) Lemma. The coefficient matrix of (45) is a Leontief matrix with substitution having the property

$$e_m A^1 > 0 \qquad \text{and} \qquad e_m T^2 > 0$$

Proof. It suffices to note that $e_m A^1 > 0$ holds by definition and that

$$e_m T^2_j = e_m(u_j - \theta^1_j) = 1 - e_m\theta^1_j > 0 \quad \text{by (21)}. ||$$

(48) Theorem. To solve the improvement subprogram (45) is equivalent to solving problem (30) when the pivot choice is restricted to the variables x^2 and y^2.

Proof. First, we note that, according to (47) and (2-6), any feasible basis to subproblem (45) is a Leontief matrix; therefore, by (18), the corresponding matrix

$$\tilde{B} = \begin{bmatrix} B^1 & \tilde{E}^2 \\ & \\ 0 & \tilde{B}^2 \end{bmatrix}$$

is a feasible basis to problem (30). Next, we observe that, by (44), no column of B^1 is eliminated; hence, by (32a), the simplex multipliers $\bar{\pi}^1$ associated with the first m rows of $\tilde{B}$ are constant and the simplex multipliers $\bar{\pi}^2$ associated with the last m rows of B are the simplex multipliers of (30) corresponding to the basis $\tilde{B}^2$. Consequently, the columns of (45) and the corresponding columns of (30) price out identically. This implies that the pivot columns and, therefore, the pivot rows, are the same, and since there is a one to one correspondence between the bases $\tilde{B}$ and $\tilde{B}^1$ the conclusion of the theorem follows. ||

The above theorem tells us that by solving subproblem (45) we find an improved basis to the original problem (1). To this basis correspond the simplex multipliers (33)

$$\pi^1 = \bar{\pi}^1 + \bar{\pi}^2 \theta^1$$

$$\pi^2 = \bar{\pi}^2$$

with which the coefficient columns of the variables x^2 and y^2 price out nonnegative. If the coefficient columns of the variables x^1 and y^1

also price out nonnegative, then the basis thus obtained is optimal for problem (1). If these columns do not price out nonnegative, then, to further improve the feasible solution, we reverse the decomposition step just described by transforming, this time, the coefficient matrix of problem (1) by

$$P^2 = \begin{bmatrix} I_m & \theta^2 \\ 0 & I_m \end{bmatrix}$$

where θ^2 corresponds to the optimal basis found by solving (45). Thus we obtain the new improvement subprogram

$$(49) \quad \begin{cases} \text{Minimize } Z^1 \text{ satisfying} \\ A^1x^1 + T^1y^1 = b^1 \\ c^1x^1 + t^1y^1 = Z^1 \text{ (min)} \\ \quad x^1 \geq 0, \quad y^1 \geq 0 \end{cases}$$

where

$$T^1 = (I_m - \theta^2)_{\bar{I}^2} \qquad \text{and} \qquad \bar{I}^2 = M - I^2$$

$$t^1 = (d^1 + \bar{\pi}^2)_{\bar{I}^2} \quad \text{where this time } \bar{\pi}^2 \text{ is defined by}$$

(32b)

It is obvious now that, by solving iteratively the two subproblems (45) and (49) we will be lead in a finite number of steps (a consequence of (46)) to the optimal solution of problem (1). It remains to indicate that the optimal solution to problem (1) is given by (38).

Finally, to summarize, we illustrate in figure 3-1 below the steps of the decomposition procedure just described. The following is the key to the numbers in parentheses appearing in the figure.

(1) coefficient matrix of problem (1)

(2) feasible basis B(0) to problem (1)

(3) basis to equivalent problem

(4) improvement subprogram a(1) for which a feasible basis is $\overline{B}^2(0)$

(5) optimal basis to a(1)

(6) improved feasible basis to problem (1)

(7) basis to equivalent problem

(8) improvement subprogram b(1) for which a feasible basis is $\overline{B}^1(0)$

(9) optimal basis to problem (1)

(10) improved feasible basis to problem (1)

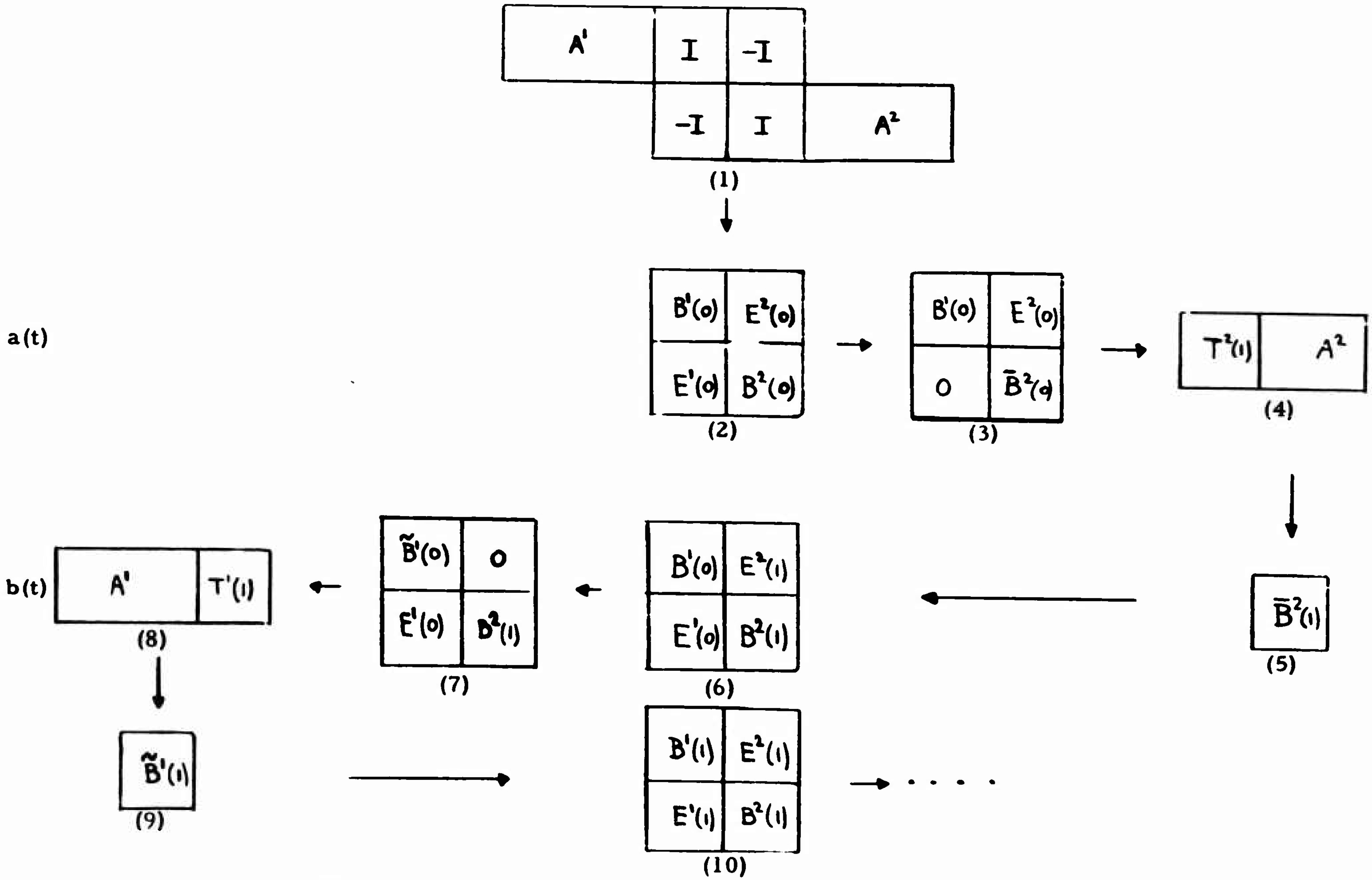

Fig. 3-1. Decomposition by square block triangularization.

3-5. Algorithm.

The algorithm for solving problem (1) which is based on the decomposition method outlined in section 3 will solve iteratively the subproblems

$$
a(t) \begin{cases} \text{Minimize } Z^1 \text{ subject to} \\ A^1x^1 + T^1y^1 = e_m \\ c^1x^1 + t^1y^1 = Z^1 \text{ (min)} \\ x^1 \geq 0, \quad y^1 \geq 0 \end{cases}
$$

and

$$
b(t) \begin{cases} \text{Minimize } Z^2 \text{ subject to} \\ A^2x^2 + T^2y^2 = e_m \\ c^2x^2 + t^2y^2 = Z^2 \text{ (min)} \\ x^2 \geq 0, \quad y^2 \geq 0 \end{cases}
$$

where

A^1 and A^2 are the Leontief matrices with substitution of problem (1);

c^1 and c^2 are the cost vectors of problem (1);

e_m is an m-component vector whose elements are all one.

T^1, t^1, T^2, t^2 will be defined in steps 3 and 4 below.

We now list the steps of the algorithm.

Step 1. Solve the subproblem

$$a(1)\begin{cases}\text{Minimize } Z^1 = c^1x^1 \\ \text{subject to } A^1x^1 = e_m, \quad x^1 \geq 0\end{cases}$$

This yields an optimal basis $\overline{B}^1(1)$, its inverse $\beta^1(1)$ and optimal simplex multipliers $\tilde{\pi}^1(1)$. Set $I^1(1) = \varphi$ and $\theta^1(1) = 0$ (zero matrix). Set $\overline{\pi}^1(1) = \tilde{\pi}^1(1)$. Go to step 2.

Step 2. Solve the subproblem

$$b(1)\begin{cases}\text{Minimize } Z^2 = c^2x^2 + [d^2 + \overline{\pi}^1(1)]\, y^2 \\ \text{subject to } A^2x^2 + I_m y^2 = e_m \\ \qquad x^2 \geq 0, \quad y^2 \geq 0\end{cases}$$

This yields an optimal basis $\overline{B}^2(1)$, its inverse $\overline{\beta}^2(1)$, optimal simplex multipliers $\overline{\pi}^2(1)$ and an importation set $I^2(1)$. Let $\tilde{\pi}^2(1) = \pi^2(1)$ and $\theta^2(1) = - E^2(1)\overline{\beta}^2(1)$ (14)*

Set $t = 2$ and go to step 3.

Step 3. Solution of the improvement subprogram $a(t)$. (49)

The following data from $b(t-1)$ are needed:

---

*This and the subsequent numbers in parentheses refer to equations given in preceding sections.

$I^2 = I^2(t-1)$, $\theta^2 = \theta^2(t-1)$ and $\bar{\pi}^2 = \bar{\pi}^2(t-1)$.

(a) Compute the initial simplex multipliers of $a(t)$

$$\tilde{\pi}^1 = \bar{\pi}^1(t-1) + \bar{\pi}^2\theta^1(t-1) \qquad (34a)$$

and also, if $\tilde{\pi}^2(t-1)$ is not easily available,

$$\tilde{\pi}^2 = \bar{\pi}^2 - \tilde{\pi}^1\theta^2 = \tilde{\pi}^2(t-1) \qquad (34a)$$

(b) Set up the coefficient matrix of $a(t)$ with

$$T^1 = [I_m - \theta^2]_{\bar{I}^2} \qquad \text{where } \bar{I}^2 = M - I^2$$

$$t^1 = [d^1 + \tilde{\pi}^2]_{\bar{I}^2} \qquad (50)$$

(c) Compute the inverse

$$\bar{\beta}^1 = \beta^1(t-1)[I - \theta^2\theta^1(t-1)]^{-1} \qquad (41b)$$

(d) Using $\tilde{\pi}^1$ check if the initial basis $\bar{B}^1$ is optimal. If it is, set $k = 1$ and go to step 5. If not, go to substep (e).

(e) Solve the improvement subprogram $a(t)$ starting with the initial basis $\tilde{B}^1$. This yields an optimal basis $\tilde{B}^1(t)$, as well as $I^1(t)$, $\tilde{\beta}^1(t)$ and $\tilde{\pi}^1(t)$.

(f) Compute the inverse

$$\beta^1(t) = \tilde{\beta}^1(t)[I + \theta^2\tilde{\theta}^1]^{-1} \quad \text{where } \tilde{\theta}^1 = -E^1(t)\tilde{\beta}^1(t) \qquad (42a)$$

Compute

$$\bar{\pi}^1(t) = \gamma^1\beta^1(t) \qquad (32a)$$

and set

$$\theta^1(t) = -E^1(t)\,\beta^1(t)$$

Go to step 4.

<u>Step 4.</u> Solution of the improvement subprogram b(t) .

The following data from a(t) are needed

$$I^1 = I^1(t), \qquad \theta^1 = \theta^1(t), \quad \text{and} \quad \tilde{\pi}^1 = \tilde{\pi}^1(t)$$

(a) Compute the initial simplex multipliers of b(t)

$$\overline{\pi}^2 = \tilde{\pi}^2(t-1) + \tilde{\pi}^1\theta^2(t-1) \qquad (34b)$$

and also, if $\overline{\pi}^1(t)$ is not easily available,

$$\overline{\pi}^1 = \tilde{\pi}^1 - \overline{\pi}^2\theta^1 = \overline{\pi}^1(t) \qquad (34b)$$

(b) Set up the coefficient matrix of b(t) with

$$T^2 = [I_m - \theta^1]_{\overline{I}^1} \qquad \text{where } \overline{I}^1 = M - I^1$$

$$t^2 = [d^2 + \overline{\pi}^1]_{\overline{I}^2} \qquad (46)$$

(c) Compute the inverse

$$\overline{\beta}^2 = \beta^2(t-1)[I - \theta^1\theta^2(t-1)]^{-1} \qquad (41a)$$

(d) Using $\overline{\pi}^2$ check if the initial basis $\overline{B}^2$ is optimal. If it is, set k = 2 and go to step 5. If not, go to substep (e).

(e) Solve the improvement subprogram b(t) starting with the initial basis $\overline{B}^2$. This yields an optimal basis $\overline{B}^2(t)$, as well as $I^2(t)$, $\overline{\beta}^2(t)$ and $\overline{\pi}^2(t)$.

(f) Compute the inverse

$$\beta^2(t) = \overline{\beta}^2(t)[I + \theta^1\overline{\theta}^2]^{-1} \quad \text{where } \overline{\theta}^2 = -E^2(t)\overline{\beta}^2(t) \qquad (42)$$

Compute

$$\tilde{\pi}^2(t) = \gamma^2\beta^2(t) \qquad (32b)$$

and set

$$\theta^2(t) = - E^2(t)\,\beta^2(t)$$

(g) Set $t = t + 1$ and return to step 3.

Step 5. Computation of an optimal solution. (38)

(a) if $k = 1$, then an optimal solution to problem (1) is

$$w^1 = \tilde{\beta}^1 [\, b^1 + \theta^2 b^2]$$

$$w^2 = \beta^2 [\, b^2 - E^1(t-1) w^1]$$

(b) if $k = 2$, then an optimal solution to problem (1) is

$$w^2 = \bar{\beta}^2 [\, b^2 + \theta^1 b^1]$$

$$w^1 = \beta^1 [\, b^1 - E^2(t-1) w^2]$$

This completes the statement of the algorithm.

Some remarks are in order now. First, it should be mentioned that it is possible to limit the exchange of information between the subproblems $a(t)$ and $b(t)$ to some subsets of θ^1 and θ^2, which must necessarily include the columns corresponding to I^2 and I^1 respectively, provided that some obvious precautions concerning the optimality criterion are taken. For instance one might communicate to subproblem $a(t)$ only the columns j of θ^2 for which the simplex multipliers π^1 and π^2 of the original problem satisfy the relation

$$[\, d^1 + \pi^2]_j \leq \pi^1_j \qquad j \in M.$$

We have reasons to believe that for such a selection the convergence of the preceding algorithm would not be greatly affected. The above remark should be important for the solution of problems concerning the cooperation between two economies.

Finally, we wish to indicate that, if in step 3 the matrix $\bar{\theta}^2(t-1)$ is readily available, then it will be easier to compute $\tilde{\beta}^1$ through (43). Of course, the same remark holds for step 4.

3-6. Conclusions.

Some concluding remarks are in order now. They will all have to do with comparing the preceding algorithm with the method of chapter II as well as with the decomposition principle of Dantzig and Wolfe [9, ch. 23]. We are not going to prove our assertions, but will limit ourselves to summarizing some of the observations we have made.

First, the structural parallelism of the algorithms of chapters 2 and 3 is illustrated in figure 3-2. As was already indicated, they have in common the fact that both are symmetrical decomposition procedures based on price communication between the two sub-problems. However, in the preceding algorithm some additional exchange of information is required, i. e., the exchange of the adjustment matrices θ^1 and θ^2 (28). It is this additional information which permits the "acceleration" of the convergence. Why? Simply because the simplex multipliers (π^1, π^2) corresponding to a basis of problem (1) are found directly by the square block triangularization method, whereas they are found only in the limit by the price communication method.

The following economic interpretation can be given to illustrate the preceding remark. Suppose an economy I imports from economy II one commodity i at the price $\pi_i^2(t-1)$, and suppose that, under these conditions, optimal prices for economy I are $\pi^1(t)$. Next, suppose that economy II discovers that by importing commodity j from economy I it can decrease its optimal prices $\pi^2(t-1)$ to $\pi^2(t)$. But this decrease will cause a decrease in $\pi^1(t+1)$ which in its turn will cause a new decrease in $\pi^2(t+1)$, and so on. Now, the algorithm of chapter II actually follows this iterative process, whereas the algorithm of chapter III finds the limiting prices π^1 and π^2 right away. For further clarification of this remark see (27) and (28).

The comparison with the decomposition method of Dantzig and Wolfe would require a lengthy exposition. We shall limit ourselves to the following conclusions.

(i) If the standard decomposition method is applied to the primal problem (1), then an asymmetry is introduced in the sense that one part of the program will become the master program and the other part the subprogram. Consequently, the exchange of information between both parts is not the same in both directions.

(ii) It can be proved that when the decomposition principle is applied to problem (1), it is advantageous to solve the subprogram

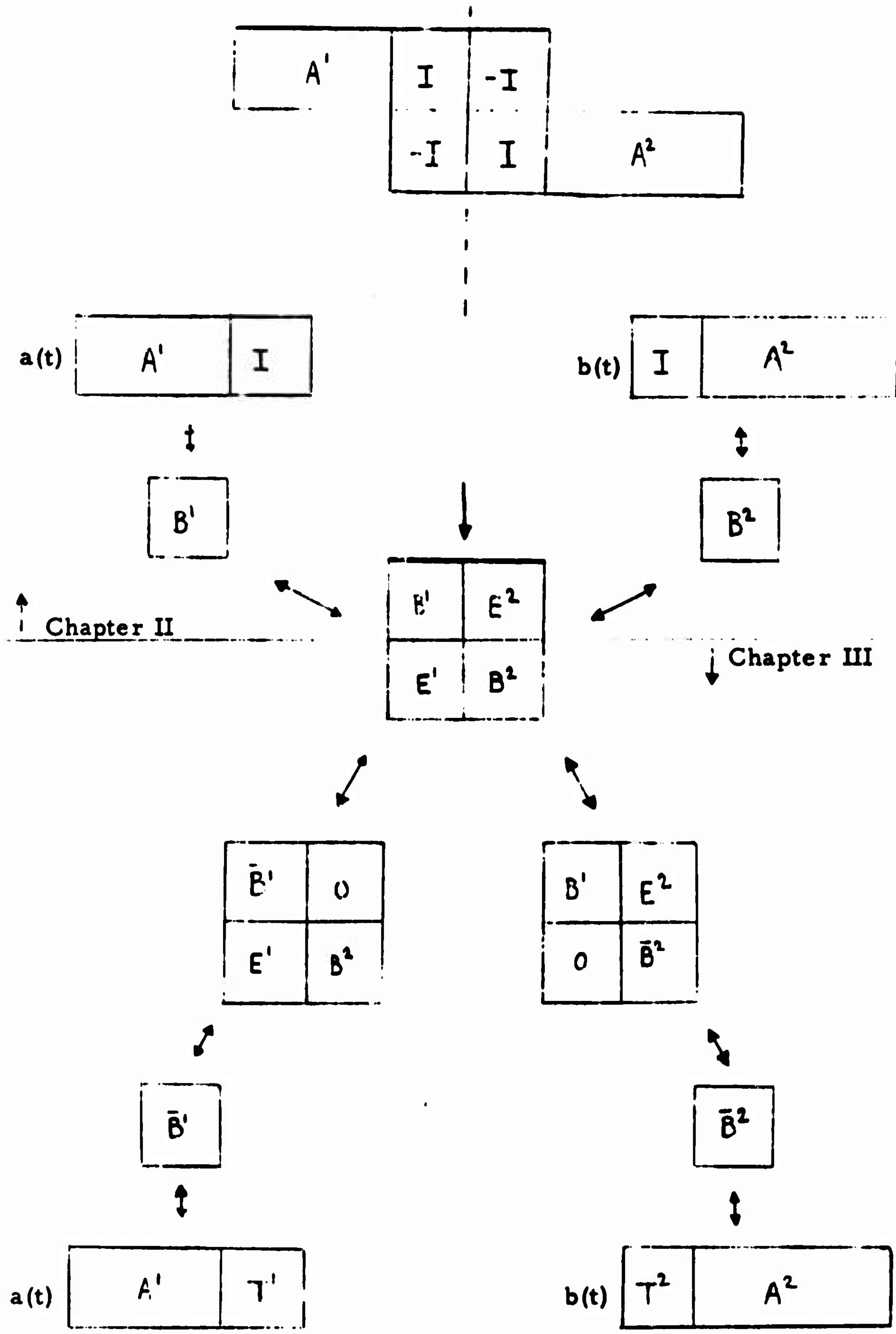

Fig. 3-2. Relations between the bases of Chapters II and III.

by adopting the following rules.

(a) First solve completely the subprogram, disregarding any homogeneous solutions which might arise.*

(b) When the optimal solution is found in (a), consider all the homogeneous solutions and introduce them in the master program.

Note. The application of rule (b) corresponds exactly to the communication of the information $-\theta^1 E^2$ of chapter III.

(iii) It can be shown that when rules (a) and (b) are applied, the master program works on the same principle as the algorithm of chapter III, whereas the subprogram has the same function as a subproblem of chapter II.

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*This contradicts the generality of the empirical statement [7, p. 11] that it is more efficient not to solve the subprogram completely.

CHAPTER IV

GENERALIZATION OF THE DECOMPOSITION METHOD BY SQUARE BLOCK TRIANGULARIZATION

4-1. Application to N Interconnected Leontief Systems.

In this section we shall see how the decomposition method by square block triangularization can be extended to solve linear programming problems based on more than two interconnected Leontief systems. Generally, these problems can be stated in detached coefficient form as follows:

(1) Minimize Z subject to $y^{ij} \geq 0$

$x^1 \geq 0$	$x^2 \geq 0$	$x^3 \geq 0$	y^{12}	y^{13}	y^{21}	y^{23}	y^{31}	y^{32}	const.
A^1			I	I	$-I$		$-I$		$b^1 \geq 0$
	A^2		$-I$		I	I		$-I$	$b^2 \geq 0$
		A^3		$-I$		$-I$	I	I	$b^3 \geq 0$
c^1	c^2	c^3	d^{12}	d^{13}	d^{21}	d^{23}	d^{31}	d^{32}	Z (min)

where we assume that

$$(2)\quad \begin{cases} e_m A^i > 0 & \text{for} \quad i = 1, 2, 3 \\ d^{ij} + d^{ji} > 0 & \text{for} \quad i \neq j \quad \text{and} \quad i = 1, 2, 3, \quad j = 1, 2, 3 \\ c^i \geq 0 & \text{for} \quad i = 1, 2, 3 \end{cases}$$

We shall first consider a special case of problem (1) (arising from serially interconnected Leontief systems) whose coefficient matrix and feasible bases have the following staircase structures:

$$(3)\ A = \begin{array}{|c|c|c|c|c|c|c|} \hline A^1 & I & -I & & & & \\ \hline & -I & I & A^2 & I & -I & \\ \hline & & & & -I & I & A^3 \\ \hline \end{array} \qquad B = \begin{array}{|c|c|c|} \hline B^1 & E^{21} & 0 \\ \hline E^{12} & B^2 & E^{32} \\ \hline 0 & E^{23} & B^3 \\ \hline \end{array}$$

where B^1, B^2 and B^3 are square Leontief matrices.

To decompose such a problem the following generalization of the method described in section 3-4 can be applied.

(a) Given a non-optimal basis B we consider the transformation matrix

$$P(a) = \begin{array}{|c|c|c|} \hline I & 0 & 0 \\ \hline \theta(a) & I & 0 \\ \hline 0 & 0 & I \\ \hline \end{array}, \qquad \text{where } \theta(a) = -E^{12}\beta^1,$$

and compute the simplex multipliers

$$\bar{\pi}^1 = \gamma^1\beta^1.$$

(b) Premultiplying by $P(a)$ the coefficient matrix A we obtain an equivalent problem whose coefficient matrix has the form

(4)
$$A(a) = \begin{array}{|c|c|c|c|c|c|c|} \hline A^1 & I & I & & & & \\ \hline \bar{A}^1 & -\bar{T}^1 & \bar{T}^1 & A^2 & I & -I & \\ \hline & & & & -I & I & A^3 \\ \hline \end{array}$$

where

$$\bar{A}^1 = \theta(a) A^1$$

$$\bar{T}^1 = I - \theta(a) .$$

We note that the corresponding feasible basis is

(5)
$$B(a) = P(a) B = \begin{array}{|c|c|c|} \hline B^1 & E^{21} & 0 \\ \hline 0 & B^2(a) & E^{32} \\ \hline 0 & E^{23} & B^3 \\ \hline \end{array}$$

where

$$B^2(a) = B^2 + \theta(a) E^{21}$$

(c) Next, consider the following subproblem:

Minimize Z^1 subject to:

(6)

$x^{21} \geq 0$	$x^2 \geq 0$	$y^{23} \geq 0$	$y^{32} \geq 0$	$x^3 \geq 0$	Constants
T^{21}	A^2	I	-I	0	e_m
		-I	I	A^3	e_m
t^{21}	c^2	d^{23}	d^{32}	c^3	Z^1 (min)

where*

$$x^{21} = [y^{21}]_{\bar{I}^{12}} \qquad \text{and} \qquad \bar{I}^{12} = M - I^{12}$$

$$T^{21} = [I - \theta(a)]_{\bar{I}^{12}}$$

$$t^{21} = [d^{21} + \bar{\pi}^{1}]_{\bar{I}^{12}}$$

We note that this subproblem is the problem studied in chapter III, because, according to III-(24), the matrix T^{21} has the property

$$e_m T^{21} > 0;$$

hence, subproblem (6) can be solved, starting with the basis

$$\bar{B}_a = \begin{array}{|c|c|} \hline B^2(a) & E^{32} \\ \hline E^{23} & B^3 \\ \hline \end{array}$$

by the algorithm of section 3-5.

(d) It can be easily verified that to an optimal basis of subproblem (6) corresponds a feasible basis of the original problem. If this basis is not optimal, then we repeat the preceding step by transforming, this time, B into

---

*By definition II-(22), I^{12} is the importation set $\{j/E_k^{12} = -u_j\}$ which corresponds to the basic variables y_j^{12}.

$$B(b) \;=\; P(b)\,B \;=\; \begin{array}{|c|c|c|}\hline B^1 & E^{21} & 0 \\ \hline E^{12} & B^2(b) & 0 \\ \hline 0 & E^{23} & B^3 \\ \hline \end{array}$$

where $B^2(b) \;=\; B^2 + \theta(b)E^{23}$ and $\theta(b) \;=\; -\,E^{32}\beta^3$

Thus, the decomposition method for solving a multistage Leontief system is a straightforward extension of the method outlined in chapter III.

We shall briefly show now that the same holds for the decomposition method for solving problem (1) in general; however, more adjustments have to be made, and this increases the necessary exchange of information between the subprograms. At this point it should be recalled that the decomposition procedure by price communication which was developed in chapter II yields very easily a feasible basis for problem (1). Therefore, in order to avoid unprofitable computations, it is suggested that this method be used to determine an improved starting basis for the decomposition method by square block triangularization whose basic steps are:

(a) transform the starting feasible basis as follows:

(7) $\qquad B(a) \;=\; P(a)\,B \;=$

$$\begin{array}{|c|c|c|}\hline I & & \\ \hline \theta^{12} & I & \\ \hline \theta^{13} & & I \\ \hline \end{array} \;\times\; \begin{array}{|c|c|c|}\hline B^1 & E^{21} & E^{31} \\ \hline E^{12} & B^2 & E^{32} \\ \hline E^{13} & E^{23} & B^3 \\ \hline \end{array} \;=\; \begin{array}{|c|c|c|}\hline B^1 & E^{21} & E^{31} \\ \hline 0 & B^2(a) & E^{32}(a) \\ \hline 0 & E^{23}(a) & B^3(a) \\ \hline \end{array}$$

where

$$\theta^{12} = -E^{12} \qquad \theta^{13} = -E^{13}$$

$$B^{2}(a) = B^{2} + \theta^{12}E^{21} \qquad B^{3}(a) = B^{3} + \theta^{13}E^{31}$$

$$E^{23}(a) = E^{23} + \theta^{13}E^{21} \qquad E^{32}(a) = E^{32} + \theta^{12}E^{31}$$

(b) note that the improvement program has the form

A^{1}			I	I	$-I$		$-I$	
$\bar{A}^{1}$	A^{2}		$-\bar{T}^{21}$	θ^{12}	$\bar{T}^{21}$	I	$-\theta^{12}$	$-I$
$\bar{\bar{A}}^{2}$		A^{3}	θ^{13}	$-\bar{T}^{31}$	$-\theta^{13}$	$-I$	$\bar{T}^{31}$	I

where $\bar{T}^{21} = I - \theta^{12}$ and $\bar{T}^{31} = I - \theta^{13}$.

(c) Improve the basic solution of the original problem (1) by solving the following improvement subprogram [see III-(48)].

(8) Minimize Z subject to

$x^{21} \geq 0$	$x^{2} \geq 0$	$y^{23} \geq 0$	$y^{32} \geq 0$	$x^{3} \geq 0$	$x^{31} \geq 0$	constants
T^{21}	A^{2}	I	$-I$	0	$-\bar{\theta}^{12}$	e_m
$-\bar{\theta}^{13}$	0	$-I$	I	A^{3}	T^{31}	e_m
t^{21}	c^{2}	d^{23}	d^{32}	c^{3}	t^{31}	Z (min)

where

$$x^{21} = (y^{21})_{\bar{I}^{12}} \qquad x^{31} = (y^{31})_{\bar{I}^{13}}$$

$$T^{21} = (I - \theta^{21})_{\bar{I}^{12}} \qquad T^{31} = (I - \theta^{31})_{\bar{I}^{13}}$$

$$t^{21} = (d^{21} + \bar{\pi}^{1})_{\bar{I}^{12}} \qquad t^{31} = (d^{31} + \bar{\pi}^{1})_{\bar{I}^{13}}$$

$$\bar{\theta}^{12} = (\theta^{12})_{\bar{I}^{21}} \qquad \bar{\theta}^{13} = (\theta^{13})_{\bar{I}^{13}}$$

(d) Finally, note that the coefficient matrix of (8) is a Leontief matrix. Therefore, a decomposition procedure similar to the algorithm of section 3-5 can be applied to solve this subproblem. It can be verified without difficulty that the final subproblem to be solved will have the form:

$$(9) \qquad \begin{cases} \text{Min } Z^3 \text{ subject to} \\ T^{32}x^{23} + A^3x^3 + T^{31}x^{31} = e_m \\ t^{32}x^{23} + c^3x^3 + \bar{t}^{31}x^{31} = Z^3\,(\text{min}) \end{cases}$$

where

$$T^{32} = [I - \theta^{32}]_{\bar{I}^{23}} \qquad \theta^{32} = -E^{23}\beta^2(a)$$

$$(10) \qquad t^{32} = [d^{32} + \bar{\pi}^2]_{\bar{I}^{23}} \qquad \bar{\pi}^2 = \bar{\gamma}^2\beta^2(a)$$

$$\bar{t}^{31} = [t^{31} + \bar{\theta}^{12}\bar{\pi}^2]$$

4-2. General Two-Stage Problem.

In this section we shall extend the decomposition method by square block triangularization to the general two-stage problem

$$
(11)\qquad \begin{cases}
\text{Minimize } Z \text{ subject to} \\
A^1x^1 + I_m y^1 - I_m y^2 \hphantom{+ A^2x^2} = b^1 \geq 0 \\
\qquad - I_m y^1 + I_m y^2 + A^2x^2 = b^2 \geq 0 \\
c^1x^1 + d^1y^1 + d^2y^2 + c^2x^2 = Z\ (\min) \\
x^1 \geq 0, \quad y^1 \geq 0, \quad y^2 \geq 0, \quad x^2 \geq 0
\end{cases}
$$

where A^1 and A^2 are arbitrary matrices of size $(m \times n_1)$ and $(m \times n_2)$ respectively, and d^1 and d^2 satisfy the condition

$$
(12)\qquad d^1 + d^2 > 0
$$

which, as indicated before, is necessary for a solution to problem (11) to exist.

Basically, the method to be described will be similar to the one outlined in section 3-4. However, in the present case, a complication arises at each simplex step of the improvement subprograms because we have to take into account the possibility that the new corresponding basis to problem (11) might become infeasible. The main object of this section is to show how this difficulty can be overcome by an additional exchange of information requiring only simple computations.

We begin by generalizing our notations. To do this we must consider a feasible basis of problem (11) which, generally, has the form

$$\tilde{B} = \begin{bmatrix} \tilde{B}^1 & \tilde{E}^2 \\ \tilde{E}^1 & \tilde{B}^2 \end{bmatrix} = \left[\begin{array}{c|c|c|c} D^1 & \begin{matrix} 1 & \\ & 1 \end{matrix} & \begin{matrix} -1 & \\ & -1 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -1 & \\ & -1 \end{matrix} & \begin{matrix} +1 & \\ & +1 \end{matrix} & D^2 \end{array}\right]$$

where the submatrices $\tilde{B}^1$ and $\tilde{B}^2$ <u>are not necessarily square.</u> We shall first show that

(14) <u>Lemma.</u> <u>It is always possible to transform B, by column</u> permutations only, into the matrix

$$B = \begin{bmatrix} B^1 & E^2 \\ E^1 & B^2 \end{bmatrix} = \left[\begin{array}{c|c|c|c} D^1 & \begin{matrix} 1 & \\ & -1 \end{matrix} & \begin{matrix} 1 & \\ & -1 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -1 & \\ & 1 \end{matrix} & \begin{matrix} -1 & \\ & 1 \end{matrix} & D^2 \end{array}\right]$$

where B^1 is a <u>nonsingular</u> submatrix of order m.

<u>Proof.</u> Let the submatrix $[\tilde{B}^1, \tilde{E}^2]$ of B be denoted by $[D, H, 0]$, where H corresponds to the connection columns, and let us suppose that a basis for this submatrix is

$$\overline{B}^1 = [\overline{D}, \overline{H}]$$

where $\overline{D}$ and $\overline{H}$ are submatrices of D and H respectively. We first note that the nonsingularity of B implies that:

(i) $\overline{B}^1$ has rank m

(ii) the columns of D are linearly independent

(iii) the columns of H are linearly independent.

Next, expressing a nonbasic column D_j in terms of $\overline{B}^1$, i.e.,

$$D_j = \overline{D}\lambda^1 + \overline{H}\lambda^2$$

we note that, since (ii) holds, there must exist at least one component of λ^2, say λ_k^2, which is nonzero. Hence, we can replace in the basis $\overline{B}^1$ the column H_k by the column D_j and thus obtain a new basis. By repeating this operation we can introduce all the columns of D into the basis. ||

Thus, B of lemma (14) becomes the counterpart of the Leontief matrix III-(14); accordingly, we generalize the definitions of the importation sets II-(22) as follows:

$$
(15) \qquad \begin{aligned} I^1 &= \{i / E_j^1 = \pm u_i\} \quad \text{and} \\ I^2 &= \{i / E_j^2 = \pm u_i\} \end{aligned}
$$

and we note that, since B is nonsingular, we must have

$$
(16) \qquad I^1 \cap I^2 = \varphi
$$

We turn now to the steps of the decomposition method to solve problem (11). To do this we shall follow the development given in section 3-4 and refer to it whenever the situations are parallel.

We start by assuming that we have a feasible basis to problem (11) which has the form (14). Under this condition, the inverse of B^1 exists and therefore, as before, we can transform problem (11) into an equivalent problem which can be stated as

$$
(17) \qquad \begin{cases} \text{Minimize } Z = c^1x^1 + d^1y^1 + d^2y^2 + c^2x^2 \\ \text{subject to} \quad A^1x^1 + I_m y^1 - I_m y^2 = b^1 \\ \qquad \bar{A}^1x^1 - \bar{T}^2y^1 + \bar{T}^2y^2 + A^2x^2 = \bar{b}^2 \\ \qquad x^1 \geq 0, \quad y^1 \geq 0, \quad y^2 \geq 0, \quad x^2 \geq 0 \end{cases}
$$

where

$$
(18) \qquad \begin{aligned} \bar{A}^1 &= \theta^1 A^1 \\ \bar{T}^2 &= [I_m - \theta^1] \\ \bar{b}^2 &= b^2 + \theta^1 b^1 \end{aligned}
$$

and

$$\theta^1 = -E^1\beta^1$$

We recall that a feasible basis to this problem is

(19) $$\overline{B} = \begin{bmatrix} B^1 & E^2 \\ 0 & \overline{B}^2 \end{bmatrix} \qquad \text{where } \overline{B}^2 = B^2 + \theta^1 E^2$$

and we note that $\overline{B}^2$ must be nonsingular since $|\overline{B}| = |B^1| \times |\overline{B}^2| \neq 0$. Two remarks are in order here.

(i) It can be easily checked that

$$\overline{T}_j^2 = 0 \qquad \text{when} \qquad j \in I^1$$

(ii) $\overline{b}^2$ is not necessarily nonnegative, although $w^2 = \overline{\beta}^2\,\overline{b}^2 \geq 0$.

Next, we consider the following subproblem which is the improvement subprogram III-(45):

(20) $$\begin{cases} \text{Minimize } Z^2 = c^2x^2 + t^2y^2 \\ \text{subject to} \qquad A^2x^2 + T^2y^2 = \overline{b}^2 \\ \qquad\qquad x^2 \geq 0, \quad y^2 \geq 0 \end{cases}$$

where, if $\overline{I}^1 = M - I^1$ and $\pi^1 = \gamma^1\beta^1$,

$$T^2 = [I_m - \theta^1]_{\overline{I}^1}$$

$$t^2 = (d^2 + \pi^1)_{\overline{I}^1}$$

$$\bar{b}^2 = b^2 + \theta^1 b^1$$

As before, it can be easily checked that as far as the determination of the pivot column is concerned, the above subprogram is equivalent to problem (17) when the pivot choice is limited to the variables x^2 and y^2. But this is no longer true for the determination of the pivot row, since subprogram (20) disregards the first m components of the vector

$$\tilde{A}_s = \bar{B}^{-1}\bar{A}_s = B^{-1}A_s$$

where $\bar{A}_s$ is the column to be introduced into the basis $\bar{B}$ of (17). We recall that, when solving (17) by the simplex algorithm, the pivot row is determined by the criterion

$$\frac{\tilde{b}_r}{\tilde{a}_{rs}} = \min(\alpha^1, \alpha^2)$$

where, if $\tilde{b} = (\tilde{b}^1, \tilde{b}^2) = B^{-1}b$ and $\tilde{A}_s = [\tilde{A}_s^1, \tilde{A}_s^2] = B^{-1}A_s = \bar{B}^{-1}\bar{A}_s$,

$$\alpha^1 = \min_{\tilde{a}_{is}^1 > 0} \left\{ \frac{\tilde{b}_i^1}{\tilde{a}_{is}^1} \right\} \qquad (\tilde{a}_{is}^1 \equiv \tilde{A}_{is}^1)$$

(21)

$$\alpha^2 = \min_{\tilde{a}_{is}^2 > 0} \left\{ \frac{\tilde{b}_i^2}{\tilde{a}_{is}^2} \right\}$$

Furthermore, we recall that we have, by III-(37),

$$
(22) \qquad
\begin{aligned}
\tilde{A}_s^1 &= \beta^1[A_s^1 - E^2\tilde{A}_s^2] \\
\tilde{A}_s^2 &= \bar{\beta}^2[A_s^2 + \theta^1 A_s^1] = \bar{\beta}^2\bar{A}_s^2
\end{aligned}
$$

and

$$
(23) \qquad
\begin{aligned}
\tilde{b}^1 &= \beta^1[b^1 - E^2\tilde{b}^2] \\
\tilde{b}^2 &= \bar{\beta}^2[b^2 + \theta^1 b^1] = \bar{\beta}^2\bar{b}^2 .
\end{aligned}
$$

At this point it is easy to see that, as expected, α^2 is determined by the subprogram (20), whereas α^1 has to be computed separately. However, it should be noted that this can be done very easily, since all the data necessary for the computations are known. Furthermore, it should be emphasized that the exchange of information between the two systems is limited to

$$E^2\tilde{A}_s^2, \quad E^2\tilde{b}^2 \quad \text{and} \quad \alpha^1$$

This fact could become important in case of decentralized computations.

Finally, it remains to examine what has to be done when $\alpha^1 < \alpha^2$. In such a case a column corresponding to the matrix B^1 has to be eliminated from the basis B, and, therefore, a new problem (17) has to be set up which will require the following operations:

(a) determination of a new basis B^1;

(b) computation of its inverse β^1;

(c) computation of $\overline{\pi}^1 = \gamma^1 \beta^1$;

(d) modification of the coefficient matrix of subproblem (20);

(e) computation of the new inverse $\overline{\beta}^2$.

This seems quite formidable, but actually is not, as we shall see now.

If we assume that the r^{th} column is eliminated from B^1, then, to determine a new nonsingular $\tilde{B}^1$ and to compute its inverse $\tilde{\beta}^1$ we apply corollary III-(40) as follows:

(i) If $(\beta^1 A^1_s)_r \neq 0$, then we replace column B^1_r by A^1_s and compute the new $\tilde{\beta}^1$ by (24) given below, with $\overline{P}_s = \beta^1 A^1_s$. (Note that this vector was computed in (22)).

(ii) If $(\beta^1 A^1_s)_r = 0$, then, according to lemma (14), we can replace the column B^1_r by a column E^2_k for which $\beta^1_{rk} \neq 0$: we let $\overline{P}_s = (\beta^1 E^2_k) = \pm \beta^1_k$ and compute $\tilde{\beta}^1$ by

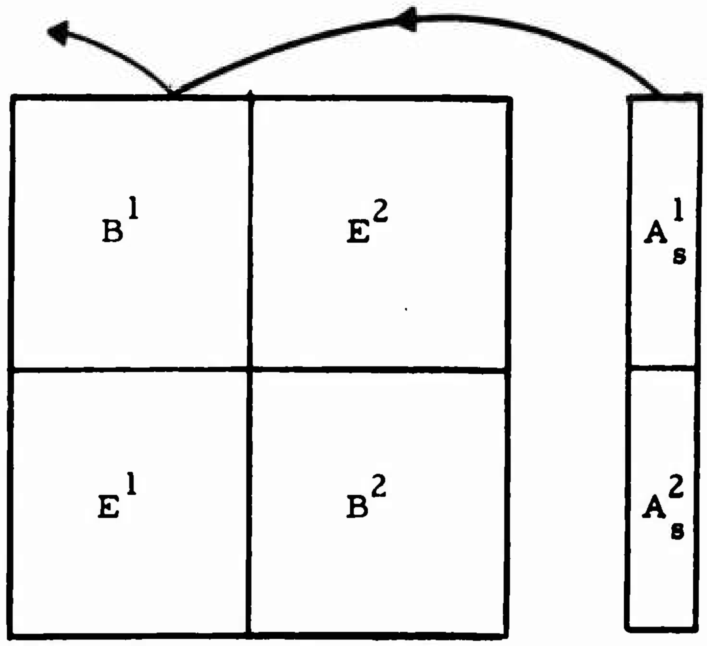
$$\tilde{\beta}^1 = \beta^1 - \frac{1}{(\overline{P}_s)_r} (\overline{P}_s - u_r) \beta^1_r \tag{24}$$

where B^1_r is the r^{th} row of β^1.

The column permutations of these operations are summarized in figure 4.1 below.

Now it is easy to carry out (c) and (d). We denote the new $\overline{\pi}^1$ by $\tilde{\pi}^1$ and the new θ^1 by $\tilde{\theta}^1$. It remains to compute the new $\overline{\beta}^2$

(i) When $(\beta^1 A^1_s)_r \neq 0$, then replace B^1_r by A^1_s



(ii) When $(\beta^1 A^1_s)_r \quad 0$ and $\beta^1_{rk} \neq 0$, then replace E^2_k and E^2_k by A^1_s

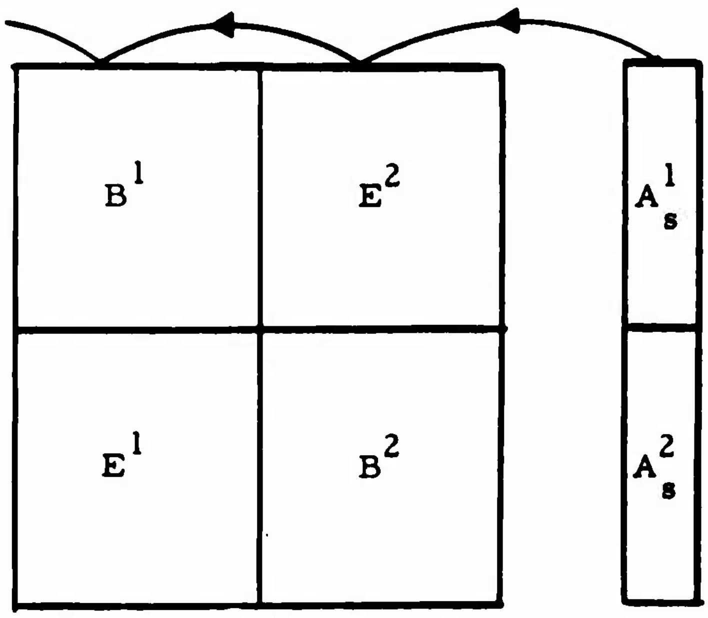


Fig. 4.1 Column elimination procedure required in order to keep B^1 nonsingular.

which will be denoted by $\bar{\bar{\beta}}^2$. Again, we have to distinguish two cases corresponding to (i) and (ii) above.

(i) If no column is replaced in B^2, then by a formula similar to III-(41) we have

$$\bar{\bar{\beta}}^2 = \bar{\beta}^2[I - (\tilde{\theta}^1 - \theta^1)\bar{\theta}^2]^{-1} \tag{25}$$

where

$$\bar{\theta}^2 = -E^2\bar{\beta}^2$$

(ii) If the k^{th} column of B^2 is replaced by A_s^2, then

$$\bar{\bar{B}}^2 = \bar{B}^2 + [\tilde{\theta}^1 - \theta^1]E^2 + [\bar{A}_s^2 - \bar{B}_k^2]\ u_k^1$$

where $\bar{A}_s^2 = \bar{\beta}^2 A_s^2$. In this case again we can first apply (25) to obtain the inverse of $\bar{B}^2 + [\tilde{\theta}^1 - \theta^1]E^2$ and then (24) (with $\bar{P}_s = \bar{A}_s^2$) to compute $\bar{\bar{\beta}}^2$.

At this point it must be emphasized that the above computations are straightforward since almost all the required data have been made available by preceding computations. However, it must be noted that the amount of computations required by the inversion of $[I - (\tilde{\theta}^1 - \theta^1)]$. in formula (25), increases with the size of the set I^1, and, therefore, the above procedure is advantageous only when I^1 is small.

From now on the decomposition method follows exactly the steps outlined in section 3-4. However, a last difficulty might arise when we switch from one improvement subprogram to the other, because the matrix B^2 corresponding to the optimal basis $\overline{B}^2$ of subprogram a(t) of section 3-5 is not necessarily nonsingular.* If B^2 is singular, then the first step to be taken is to find, by column permutations only, a nonsingular B^2. To do this we can again conveniently use Sherman-Morrison's formula [III-(40)].

To conclude this discussion we now give a <u>summary</u> of the main steps of the improvement subprogram based on the matrix A^2.

Assumption. We start with a feasible basis B for problem (11) which has the form (14).

Step 1. Compute β^1 and $\overline{\pi}^1$.

Step 2. Set up the coefficient matrix of subproblem (20).

Step 3. Compute $\overline{\beta}^2$ and $\overline{\pi}^2$.

Step 4. Determine the pivot column A_s. (If the columns price out nonnegative, go to step 6.)

Step 5. (i) If $\alpha^2 \leq \alpha^1$, introduce $\overline{A}_s^2$ into $\overline{B}^2$ and return to step 3.

(ii) If $\alpha^2 > \alpha^1$, then determine a new B^1 by using (24) and go back to step 1.

---

*It can be proved that there exists at least one ordering of the columns of the basis $\widetilde{B}$ such that the matrix B, given by (14), has the property that B^1 and B^2 are nonsingular. However, at the time of the writing of this report it is not too clear to the author whether an attempt should be made to keep B in this ideal form throughout the simplex steps of the decomposition method.

Step 6. Optimality is reached for this improvement subprogram. If the basic solution has not been improved by this subprogram, then optimality has been reached for the original problem. If it has been improved, find a matrix B^2 which is nonsingular and go to the improvement subprogram based on A^1.

Remark. This algorithm can be generalized without any difficulty to the case of N-stage problems.

BIBILIOGRAPHY

1. Abadie, J. M., "Un Nouvel Algorithme pour les Programmes Linéaires: Le Principe de Décomposition de Dantzig et Wolfe," Revue Francaise de Recherche Operationelle, Vol. 15, 1960.

2. Abadie, J. M., "The Dual Decomposition Method for Linear Programming," Computing Center, Case Institute of Technology, July 1962.

3. Abadie, J. M., "On Decomposition Principle," Operations Research Center, University of California, Berkeley, ORC 63-20 (RR), 20 August 1963.

4. Abadie, J. M., and Williams, A. C., "Dual and Parametric Methods in Decomposition," in [25], 149-158.

5. Baumol, W. J. and Fabian, T., "Decomposition, Pricing for Decentralization and External Economics," Management Science, Vol. 11, No. 1 (Sept. 1964), 1-31.

6. Beale, E. M. L., "The Simplex Method Using Pseudo-basic Variables for Structured Linear Programming Problems," in [25], 133-158.

7. Bell, E. J., "Primal-Dual Decomposition Programming," Operations Research Center, University of California, Berkeley, WP-18, 10 December 1964.

8. Bennett, J. M., "An Approach to Some Structured Linear Programming Problems," Basser Computing Department, School of Physics, The University of Sydney, March 1963.

9. Dantzig, G. B., Linear Programming and Extensions, Princeton University Press, Princeton, N. J., 1963.

10. Dantzig, G. B., "Upper Bounds, Secondary Constraints, and Block Triangularity in Linear Programming," Econometrica, Vol. 23, No. 2, April 1955, 174-183.

11. Dantzig, G. B., "Optimal Solution of a Dynamic Leontief Model with Substitution," Econometrica, Vol. 23, July 1955, 295-302.

12. Dantzig, G. B., "On the Status of Multistage Linear Programs," Management Science, Vol. 6, No. 1, October 1959.

13. Dantzig, G. B., "Compact Basis Triangularization for the Simplex Method," in [25], 125-132.

14. Dantzig, G. B., and Wolfe, P., "Decomposition Principle for Linear Programs," Operations Research, Vol. 8, No. 1, February 1960, 101-111.

15. Dantzig, G. B., and Wolfe, P., "The Decomposition Algorithm for Linear Programs," Econometrica, Vol. 29, No. 4, October 1961.

16. Dantzig, G. B., and Wolfe, P., "Linear Programming in a Markov Chain," Operations Research, Vol. 10, No. 5, October 1962.

17. Dantzig, G. B., and Wets R., "Leontief Matrices with Substitution," Notes on Operations Research - 1, Operations Research Center, University of California, Berkeley, ORC 63-19 (RN), July 1963, 16-26.

18. Dantzig, G. B., and Van Slyke, R. M., "Generalized Upper Bounding Techniques for Linear Programming," Operations Research Center, University of California, Berkeley, I, ORC 64-17, 5 August 1964 and II, ORC 64-18, February 1965.

19. Ford, L. R., and Fulkerson, D. R., Flows in Networks, Princeton University Press, Princeton, N. J., 1962.

20. Frisch, R., "The Multiplex Method for Linear Programming," Sankhya, 1957.

21. Gale, D., The Theory of Linear Economic Models, McGraw-Hill Book Company, New York, 1960.

22. Gass, S. I., Linear Programming: Methods and Applications, McGraw-Hill Book Company, New York, 1958.

23. Gass, S. I., "Dualplex Method for Large Scale Linear Programs," Thesis, University of California, Berkeley, 1965.

24. Gauthier, J. M., and Genuys, F., "Expériences sur le Principe de Décomposition des Programmes Linéaires," 1st Congress of AFCAL, Grenoble, France, September 1960, Gauthier Villars 1961, 373-381.

25. Graves, R. L., and Wolfe, P. (ed.), Recent Advances in Mathematical Programming, McGraw-Hill Book Company, New York, 1963.

26. Hadley, G., Linear Programming, Addison-Wesley Publishing Company, 1961.

27. Harvey, R. P., "The Decomposition Principle for Linear Programs," Int. J. Comp. Math., May 1964, 20-35.

28. Jaromir, A., "The Multiplex Method and its Applications to Concave Programming," Chekhoslovatskii Matematicheskii, 1962, 325-345.

29. Malinvaud, E., "Decentralized Procedures for Planning," Center for Research in Management Science, University of California, Berkeley, Technical Report No. 15, November, 1963.

30. Marcus, M., and Mine, H., A Survey of Matrix Theory and Matrix Inequalities, Allyn and Bacon, Inc., Boston, 1964.

31. Morgenstern, O. (ed.), Economic Activity Analysis, John Wiley and Sons, New York, 1954.

32. Radner, R., Notes on the Theory of Economic Planning, Center of Economic Research, Athens, Greece, 1963.

33. Rosen, J. B., "Convex Partition Programming," in [25], 159-176.

34. Rosen, J. B., "Primal Partition Programming for Block Diagonal Matrices," Numerische Mathematik, Vol. 6, 1964, 250-260.

35. Sherman, J., and Morrison, W. J., "Adjustment of an Inverse

Matrix Corresponding to a Change in One Element of a Given Matrix," The Annals of Mathematical Statistics, Vol. 21, 1950, 124-127.

36. Williams, A. C., "A Treatment of Transportation Problems by Decomposition," J. Soc. Ind. Appl. Math., Vol. 10, No. 1, March 1962, 35-48.

37. Woodbury, M. A., "Properties of Leontief Type Input-Output Matrices," in [31].

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3 REPORT TITLE

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4 DESCRIPTIVE NOTES (Type of report and inclusive dates)

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5 AUTHOR(S) (Last name, first name, initial)

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13 ABSTRACT

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